# Euclidean Shortest Path Algorithm for Spherical Obstacles

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**Abstract.** One version of the Euclidean shortest path problem (ESP) is to find the shortest path such that it starts at $p$ and ends at $q$ and it avoids passing through an unordered set of pairwise disjoint obstacles. This paper describes an approximate algorithm for solving this ESP problem for two points $p$ and $q$ and a finite set of pairwise disjoint spheres in $\mathbb{R}^3$ not containing those two points. We apply the Agarwal et al. [1] algorithm for the computation of an approximate shortest path in the free space between pairwise disjoint regular polyhedra in a preprocessing step that defines an order of the given obstacles. The resulting path is used as input for the new rubberband algorithm (RBA algorithm) that gives an approximate answer to the open question "What is the complexity of the Euclidean shortest path problem for obstacles that are disjoint balls?".
The RBA algorithm provides also a solution to the basic version of a touring-a-sequence-of-spheres problem (TSP) that finds a shortest path starting at $p$, visits all spheres in a given order and ends at $q$.
The paper discusses at first a solution for the 2-dimensional case (i.e., disks and polygons instead of spheres and polyhedra), showing that this solution extends to the 3-dimensional case.

## 1 Introduction

Many algorithms for solving various *Euclidean shortest path* (ESP) problems consider geometric objects with piecewise linear (2D or 3D) frontier segments. For example, obstacles of ESPs are modelled by polygons in 2D in [10], by polyhedra in 3D in [1, 2, 5–7, 10, 13], and by the surface of polyhedra in 2·5D in [4, 10]. Obstacles with smooth surfaces are of interest in computational geometry [11]. In this paper, we propose an algorithm for the shortest path problem avoiding sets of unordered pairwise-disjoint disks in 2D or spheres in 3D and we analyse its complexity. Our approach does not deliver an exact solution of the problem but it is a contribution for finding an answer to the open problem: "What is the complexity of the Euclidean shortest path problem for obstacles that are disjoint balls?"; see [11]. An algorithm for finding an exact solution to the general 3D ESP problem does not always exist according to Theorem 9 in [3].

## 2 Preliminaries

Let $D$ be a disk, and $p_1$ and $p_2$ be two points in $\partial D$, denoting the frontier of $D$. Let $A_1(p_1, p_2)$ and $A_2(p_1, p_2)$ be the two arcs from $p_1$ to $p_2$ in $\partial D$. We denote by $d_D(p_1, p_2)$ the length of the shorter arc $A_D(p_1, p_2)$ between $p_1$ and $p_2$.

In $\mathbb{R}^2$, a regular $m$-gon $P$ is called a *sketching* $m$-gon of a disk $D$ if $D$ is $P$'s smallest circumscribing circle.

Analogously, let $S$ be a sphere, and $p_1$ and $p_2$ be two points in $\partial S$, denoting the frontier of $S$. Let $A_S(p_1, p_2)$ be an arc from $p_1$ to $p_2$ in $\partial S$ of minimum length. We denote by $d_S(p_1, p_2)$ the length of this shortest arc $A_S(p_1, p_2)$.

In $\mathbb{R}^3$, a polyhedron $H$ is called a *sketching* polyhedron of a sphere $S$ iff each vertex of $H$ is in $\partial S$ (i.e., the surface of $S$).

For a path $\rho$, let $L(\rho)$ be its length when applying the Euclidean distance function $d_e$.

We recall some notations from [1]. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be a set of pairwise-disjoint convex polyhedral obstacles in $\mathbb{R}^d$, for $d = 2$ or $d = 3$. Let $n$ be the total number of facets of obstacles in $\mathcal{P}$. The topological closure of $\mathbb{R}^d \backslash (\cup \mathcal{P})$ is called the *free space* of $\mathcal{P}$, denoted by $\mathcal{F}(\mathcal{P})$. An *$\varepsilon$-short path* between two points $p$ and $q$ in $\mathcal{F}(\mathcal{P})$ is a path whose length is at most $(1+\varepsilon)$ times the length of the shortest path. The value $1 + \varepsilon$ is called the *approximation factor* of this approximation algorithm. We also call the path an $(1 + \varepsilon)$-*approximation* path.

We make use of the following results of [1]:

1. In $\mathbb{R}^2$, for any two points $p$, $q \in \mathcal{F}(\mathcal{P})$ and a parameter $\varepsilon$ with $0 < \varepsilon \leq 1$, an $\varepsilon$-short path between $p$ and $q$ in $\mathcal{F}(\mathcal{P})$ can be computed in $\mathcal{O}(n+(k/\sqrt{\varepsilon}) \log(k/\varepsilon))$ time.

2. In $\mathbb{R}^3$, for any two points $p$, $q \in \mathcal{F}(\mathcal{P})$ and a parameter $0 < \varepsilon \leq 1$, an $\varepsilon$-short path between $p$ and $q$ in $\mathcal{F}(\mathcal{P})$ can be computed in $\mathcal{O}(n+(k^4/\varepsilon^7) \log^3(k/\varepsilon))$ time.

## 3 Results

We describe at first an algorithm for computing an approximate Euclidean shortest path between points $p$ and $q$ visiting each disk of an ordered set of pairwise-disjoint disks in $\mathbb{R}^2$.

Later we apply this algorithm and the Agarwal et al. [1] algorithm to present an algorithm for computing an approximate Euclidean shortest path between points $p$ and $q$ avoiding a given unordered set of pairwise-disjoint disks in $\mathbb{R}^2$.

Finally we (straightforwardly) extend those algorithms for disks to algorithms that deal with an unordered set of pairwise-disjoint spheres in $\mathbb{R}^3$.

### 3.1 Algorithm in $\mathbb{R}^2$

The new algorithm for the ordered sequence of disks follows the general design principle of a *rubberband algorithm* (RBA): a set of *steps* is identified (set of arc

1: For each $i \in \{1, 2, \ldots, k\}$, let $c_i$ be the centre of $D_i$. Let vertices $p_{i-1_2}$, $p_{i_1}$ and $p_{i_2}$, $p_{i+1_1}$ be the intersection points between the line segments $c_{i-1}c_i$ and $c_ic_{i+1}$ and the disk frontiers $\partial D_{i-1}$, $\partial D_i$ and $\partial D_{i+1}$ (see Fig. 2) and let $p = c_0 = p_{0_2}$ and $q = c_{k+1} = p_{k+1_1}$. Initialize the path $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k_1}, p_{k_2}, q\rangle$.
2: Let $L_0 = +\infty$. Calculate $L_1 = d_e(p, p_{1_1}) + \sum_{i=1}^{k}(d_{D_i}(p_{i_1}, p_{i_2}) + d_e(p_{i_2}, p_{i+1_1}))$.
3: **while** $L_0 - L_1 \geq \varepsilon$ **do**
4: **for** $i = 1, 2, \ldots, k$ **do**
5: **if** $p_{i-1_2}p_{i+1_1} \cap D_i \neq \emptyset$ **then**
6: Compute tangential points $q_{i_1} \in A_D(p_{i_1}, p_{i_2})$, and $q_{i_2} \in A_D(p_{i_1}, p_{i_2})$ such that (see Fig. 3) $p_{i-1_2}q_{i_1}$ is a tangent to $A_D(p_{i_1}, p_{i_2})$ and $p_{i+1_1}q_{i_2}$ is a tangent to $A_D(p_{i_1}, p_{i_2})$
7: Update $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k_1}, p_{k_2}, q\rangle$ by replacing $p_{i_m}$ by $t_{i_m}$ in this path (where $m = 1, 2$).
8: **else**
9: Compute a point $q_i \in A_D(p_{i_1}, p_{i_2})$ such that (see Fig. 4) $d_e(p_{i-1_2}, q_i) + d_e(q_i, p_{i+1_1}) = \min\{d_e(p_{i-1_2}, p) + d_e(p, p_{i+1_1}) : p \in \partial D_i\}$
10: Update $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k_1}, p_{k_2}, q\rangle$ by replacing $p_{i_m}$ by $q_i$ in this path (where $m = 1, 2$).
11: **end if**
12: **end for**
13: Let $L_0 = L_1$. Calculate $L_1$ as in Line 2.
14: **end while**
15: Return $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k_1}, p_{k_2}, q\rangle$.

**Fig. 1.** An RBA for ordered pairwise disjoint disks.

segments that constitute an initial path between $p$ and $q$) in an initial computation. The positions of all vertices of the calculated path in the previous iteration are locally optimized one by one (by length minimization) in the current iteration. An iteration is the final one if a termination criterion is satisfied [8, 9, 12].

We start with explaining an RBA for a sequence of pairwise disjoint disks; see Fig. 1. This is a subprocedure of the main algorithm given below. The input is a set $\mathcal{P} = \{D_1, D_2, \ldots, D_k\}$ of $k$ pairwise disjoint disks given in an order, two points $p$, $q$ in free space $\mathcal{F}(\mathcal{P})$, and an accuracy constant $\varepsilon > 0$. Points $p$, $q$ and all $k$ disks are co-planar. The output is a sequence $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k_1}, p_{k_2}, q\rangle$ which starts at $p = p_0 = p_{0_1} = p_{0_2}$, then visits disks $D_i$ at points $p_{i_1}$ and $p_{i_2}$ in the given order (i.e., both points are on the frontier $\partial D_i$ and are not identical in general), and ends at $q = p_{k+1} = p_{k+1_1} = p_{k+1_2}$.

The optimal point $q_i$ in Line 9 can be computed as follows: Each point $p$ on $\partial D_i$ can be expressed as follows by $(x(t), y(t))$, where

$$x(t) = r_i \cos t + c_{ix} \quad \text{and}$$
$$y(t) = r_i \sin t + c_{iy}$$

The parameter $t$ varies from 0 to $2\pi$, $(c_{ix}, c_{iy})$ is the centre of disk $D_i$, and $r_i$ is radius of $D_i$. Let $p_{i-1} = (x_{i-1}, y_{i-1})$, and $p_{i+1} = (x_{i+1}, y_{i+1})$. We calculate

the optimal point $q_i$ by finding the optimal value of $t$ which is a solution of the following equation

$$\frac{\partial[d_e(p_{i-1}, p) + d_e(p, p_{i+1})]}{\partial t} = 0$$

Replace $p$, $p_{i-1}$, and $p_{i+1}$ by their coordinates in this equation. Applying a distance formula of basic geometry, we obtain the following equation for deciding about the optimal value of $t$:

$$\sum_{j=1}^{3}(A_j \sin jt + B_j \cos jt) + A_0 = 0 \tag{1}$$

where $A_j$, $B_j$ (for $j = 1, 2, 3$) and $A_0$ are functions of $c_{ix}$, $c_{iy}$, $r_i$, $x_{i-1}$, $y_{i-1}$, $x_{i+1}$, and $y_{i+1}$. Equation (1) can be rewritten as follows:

$$\sum_{j=0}^{6} C_j x^j = 0 \tag{2}$$

where $x = \tan \frac{t}{2}$; $C_j$ ($j = 1, 2, \ldots, 6$) are functions of $c_{ix}$, $c_{iy}$, $r_i$, $x_{i-1}$, $y_{i-1}$, $x_{i+1}$, and $y_{i+1}$.

Equation (2) can be solved in time $\mathcal{O}(1)$ using a QR algorithm of numerical linear algebra [14].

Note that the path in Fig. 2 visits each disk (TSP problem) in the given order. We only consider a precalculated subset of disks for the problem of finding the shortest path between $p$ and $q$ in the free space $\mathbb{R}^d \backslash (\cup \mathcal{D})$.

The basic idea of the main algorithm for the 2-dimensional case is quite straightforward. We employ a regular $m$-gon for approximating a disk such that

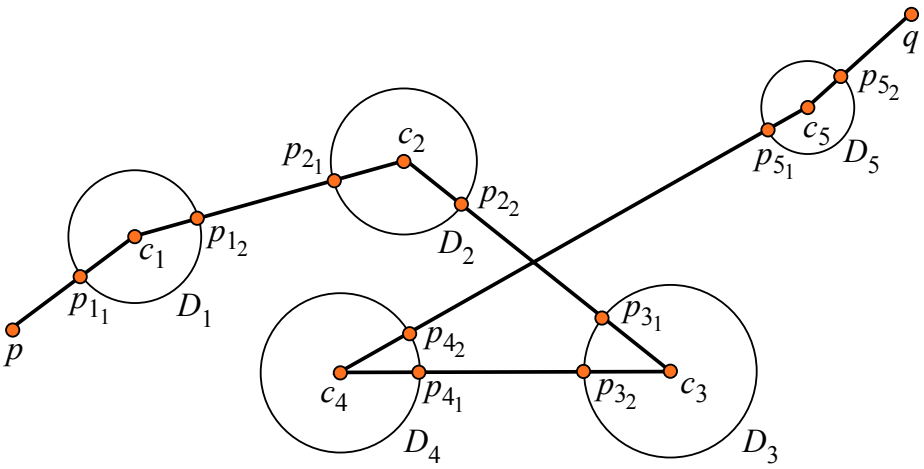


**Fig. 2.** Illustration of Line 1 in Fig. 1.

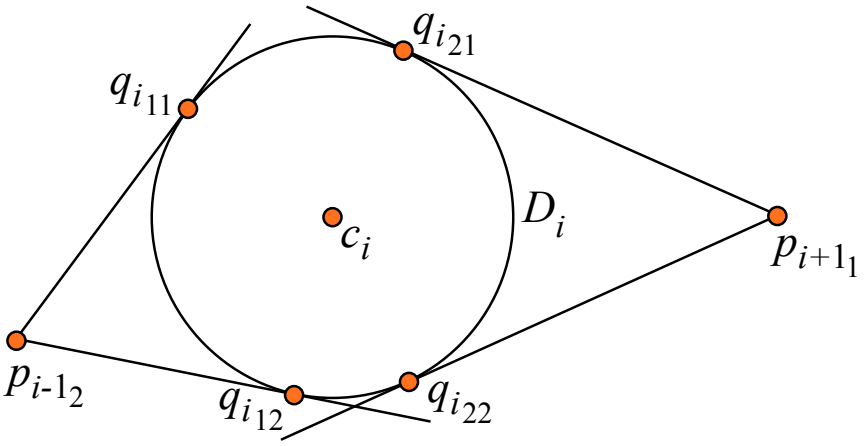


**Fig. 3.** Illustration of Line 6 in Fig. 1.

the disk's frontier is the circumscribing circle of the $m$-gon. Then we apply the Agarwal et al. algorithm in $\mathbb{R}^2$ to find an $\varepsilon_0$-short path $\rho$ between $p$ and $q$ avoiding those $k$ regular $m$-gon obstacles. This $\varepsilon_0$-short path $\rho$ delivers the initial steps (i.e., the frontiers of a set of ordered disks) for the RBA shown in Fig. 1. The RBA computes a new approximate shortest path $\rho'$ avoiding the interiors of all disks. The value $L(\rho')$ is lower bounded by the length $L(\rho) - \varepsilon_0$ that is a lower bound of the length of a true ESP between $p$ and $q$. The approximation factor of this algorithm is $L(\rho')/(L(\rho) - \varepsilon_0)$.

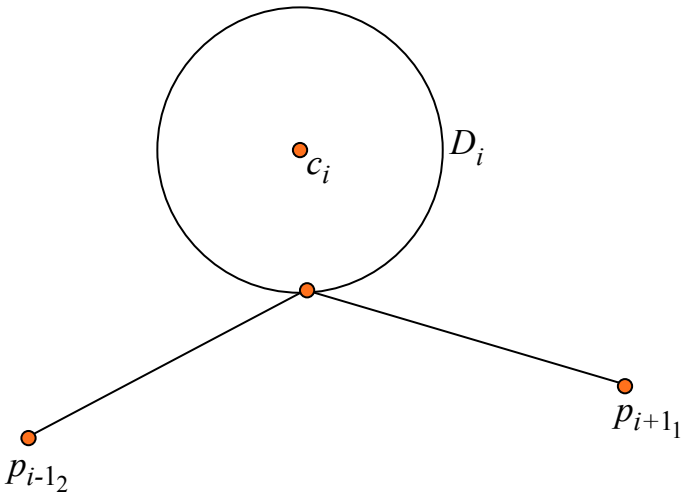


**Fig. 4.** Illustration of Line 9 in Fig. 1.

1: For each $i \in \{1, 2, \ldots, k\}$, compute a sketching $m$-gon of $D_i$.
2: Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ and $\varepsilon_0$ be inputs, apply the Agarwal et al. algorithm in $\mathbb{R}^2$ [1] (see also Section 2) to compute an $\varepsilon_0$-short path $\rho$ between $p$ and $q$.
3: Compute $\mathcal{P}' = \{P'_1, P'_2, \ldots, P'_{k'}\}$ as a sequence of regular $m$-gons such that $\rho$ starts at $p$, then crosses regular $m$-gons $P'_i$ at points $p_{i_1}$, $p_{i_2}$, ..., $p_{i_{m_i}}$ $(1 \leq m_i \leq m)$ in the given order, and finally ends at $q$.
4: Let $\mathcal{D}' = \{D'_1, D'_2, \ldots, D'_{k'}\}$ be the sequence of disks such that $D'_i$s frontier is $P'_i$s circumscribing circle.
5: Let $\mathcal{D}'$ and $\varepsilon$ be inputs for the RBA shown in Fig. 1 to compute a path $\rho'$ between $p$ and $q$.
6: Compute $\mathcal{D}'' = \{D''_1, D''_2, \ldots, D''_{k''}\}$ as the sequence of disks such that $\rho'$ starts at $p$, then intersects disks $D''_i$ in the given order, and finally ends at $q$.
7: **while** $\mathcal{D}'' \neq \mathcal{D}'$, or the value of $L(\rho')/(L(\rho) - \varepsilon_0) - 1$ is not yet small enough **do**
8: Let $\mathcal{D}' = \mathcal{D}''$.
9: Let $\mathcal{D}'$ and $\varepsilon$ be the input for the proposed RBA shown in Fig. 1 to compute the path $\rho''$ between $p$ and $q$.
10: Compute $\mathcal{D}''$ as the new sequence of disks intersected by $\rho''$.
11: Let $\rho' = \rho''$.
12: **end while**
13: Return $\rho'$.

**Fig. 5.** Main algorithm for a set of pairwise disjoint disks.

The main algorithm is shown in Fig. 5. The input is a set $\mathcal{P}$ of $k$ pairwise disjoint disks $D_1, D_2, \ldots, D_k$ in the same plane $\pi$, two points $p$, $q$ in $\mathcal{F}(\mathcal{P})$, two accuracy constants $\varepsilon > 0$ and $\varepsilon_0 > 0$, and an integer $m > 0$. The output is a sequence $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k'_1}, p_{k'_2}, q\rangle$ which starts at $p = p_0 = p_{0_1} = p_{0_2}$, then visits disks $D'_i$ at points $p_{i_1}$ and $p_{i_2}$ in the given order, and finally ends at $q = p_{k'+1} = p_{k'+1_1} = p_{k'+1_2}$.

### 3.2 Algorithm in $\mathbb{R}^3$

The basic idea of our algorithm for $\mathbb{R}^3$ is the same as for the one proposed for the 2-dimensional case.

The proposed RBA for the 3D case is shown in Fig. 6. The input is a sequence of $k$ pairwise disjoint spheres in $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$, two points $p$, $q$ in $\mathcal{F}(\mathcal{P})$, and an accuracy constant $\varepsilon > 0$. The output is a sequence $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k_1}, p_{k_2}, q\rangle$ which starts at point $p = p_0 = p_{0_1} = p_{0_2}$, then visits spheres $S_i$ at points $p_{i_1}$ and $p_{i_2}$ in the given order, and finally ends at $q = p_{k+1} = p_{k+1_1} = p_{k+1_2}$.

For Line 6 of the RBA, see Fig. 7. In this figure, $C_1$ (or $C_2$) is a cycle on $\partial S_i$ such that for each point $q_{i_{1_\alpha}}$ in $C_1$ (or $q_{i_{2_\alpha}}$ in $C_2$), $p_{i-1_2}q_{i_{1_\alpha}}$ (or $p_{i+1_1}q_{i_{2_\alpha}}$) is a tangent to $S_i$ at $q_{i_{1_\alpha}}$ (or $q_{i_{2_\alpha}}$). $C_3$ is a "largest" cycle (i.e., its centre is identical to $S_i$'s centre $c_i$ as well. Given $p_{i-1_2}$, $p_{i+1_1}$, and $S_i$, we compute $C_1$ and $C_2$ and then we compute optimal points $q_{i_{1_{\alpha_1}}}$ and $q_{i_{2_{\alpha_2}}}$ using the RBA principle.

The main algorithm is shown in Fig. 8. The inputs are spheres, two points $p$, $q$ in $\mathcal{F}(\mathcal{P})$, two accuracy constants $\varepsilon > 0$ and $\varepsilon_0 > 0$, and an integer $m > 0$. The

1: For each $i \in \{1, 2, \ldots, k\}$, let $c_i$ be the centre of $S_i$. Let initial vertex $p_{i_1}$ (or $p_{i_2}$) be the intersection point between the line segment $p_{i-1}c_i$ (or $c_i p_{i+1}$) with sphere $S_i$.
2: Let $L_0 = +\infty$. Calculate $L_1 = d_e(p, p_{1_1}) + \sum_{i=1}^{k}(d_{S_i}(p_{i_1}, p_{i_2}) + d_e(p_{i_2}, p_{i+1_1}))$.
3: **while** $L_0 - L_1 \geq \varepsilon$ **do**
4: **for** $i = 1, 2, \ldots, k$ **do**
5: **if** $p_{i-1_2}p_{i+1_1} \cap S_i \neq \emptyset$ **then**
6: Compute a point $q_{i_{l\alpha}} \in \partial S_i$, where $l = 1, 2$, $\alpha \in [0, 2\pi]$ such that (see Fig. 7) $p_{i-1_2}q_{i_{1\alpha_1}}$ is a tangent to $S_i$ at $q_{i_{1\alpha_1}}$ and $p_{i+1_1}q_{i_{2\alpha_2}}$ is a tangent to $S_i$ at $q_{i_{2\alpha_2}}$, where $\alpha_1, \alpha_2 \in [0, 2\pi]$.
7: Let $q_{i_{\alpha_1}}$ and $q_{i_{\alpha_2}}$ such that $d_{D_i}(q_{i_{\alpha_1}}, q_{i_{\alpha_2}})$ is as short as possible.
8: Update $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k_1}, p_{k_2}, q\rangle$ by replacing $p_{i_{\alpha_m}}$ in this path by $q_{i_{\alpha_m}}$ (where $m = 1, 2$).
9: **else**
10: Compute a point $q_i \in \partial S_i$ such that $d_e(p_{i-1_2}, q_i) + d_e(q_i, p_{i+1_1}) = \min\{d_e(p_{i-1_2}, p) + d_e(p, p_{i+1_1}) : p \in \partial S_i\}$
11: Update $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k_1}, p_{k_2}, q\rangle$ by replacing $p_{i_m}$ in this path by $q_i$ (where $m = 1, 2$).
12: **end if**
13: **end for**
14: Let $L_0 = L_1$. Calculate $L_1$ as in Line 2.
15: **end while**
16: Return $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k_1}, p_{k_2}, q\rangle$.

**Fig. 6.** An RBA for an ordered set of pairwise disjoint spheres.

output is a sequence of points $\langle p, p_{1_1}, p_{1_2}, p_{2_1}, p_{2_2}, \ldots, p_{k'_1}, p_{k'_2}, q\rangle$ that specifies a path with a length only an approximation factor apart from the Euclidean shortest path avoiding all spheres between $p$ and $q$.

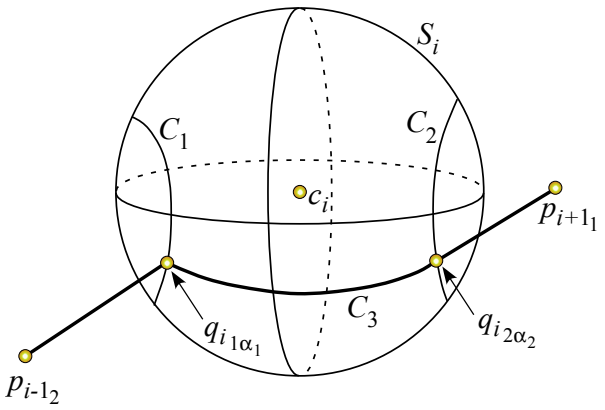


**Fig. 7.** Illustration for Line 6 in Fig. 6.

1: For each $i \in \{1, 2, \ldots, k\}$, compute a sketching polyhedra of $S_i$.
2: Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ and $\varepsilon_0$ as input, apply the Agarwal et al. algorithm in $\mathbb{R}^3$ [1] (see also Section 2) to compute an $\varepsilon_0$-short path $\rho$ between $p$ and $q$.
3: Compute $\mathcal{P}' = \{P'_1, P'_2, \ldots, P'_{k'}\}$ as a sequence of polyhedra connected by the computed path $\rho$ between $p$ and $q$.
4: Let $\mathcal{S}' = \{S'_1, S'_2, \ldots, S'_{k'}\}$ be the sequence of spheres such that $S'_i$s frontier is $P'_i$s circumscribing sphere.
5: Let $\mathcal{S}'$ and $\varepsilon$ be inputs for the proposed RBA shown in Fig. 6 to compute an initial path $\rho'$ between $p$ and $q$.
6: Compute $\mathcal{S}'' = \{S''_1, S''_2, \ldots, S''_{k''}\}$ as the sequence of spheres crossed by straight segments of path $\rho'$ between $p$ and $q$.
7: **while** $\mathcal{S}'' \neq \mathcal{S}'$, or the value of $L(\rho')/(L(\rho) - \varepsilon_0) - 1$ is not yet small enough **do**
8: Let $\mathcal{S}' = \mathcal{S}''$.
9: Let $\mathcal{S}'$ and $\varepsilon$ be inputs for the RBA shown in Fig. 6 to compute the path $\rho''$ between $p$ and $q$.
10: Compute $\mathcal{S}''$ as the new sequence of spheres.
11: Let $\rho' = \rho''$.
12: **end while**
13: Return $\rho'$.

**Fig. 8.** Main algorithm for a set of pairwise disjoint spheres.

## 4 Analysis

Regarding the time complexity of the 2-dimensional RBA shown in Fig. 1, note that the main computation is in the two stacked loops. The while-loop takes $\kappa(\varepsilon)$ iterations, where

$$\kappa(\varepsilon) = \frac{L_0 - L}{\varepsilon}$$

$L_0$ and $L$ are the lengths of initial path and output path, respectively. Lines 5–11 can be computed in $\mathcal{O}(1)$. Thus, the for-loop can be computed in time $\mathcal{O}(k)$. Thus, the algorithm can be performed in time $\kappa(\varepsilon) \cdot \mathcal{O}(k)$.

For testing the actual numerical values of $\kappa(\varepsilon)$, we have implemented a "simplified version" of the algorithm shown in Fig. 1 where all disks were degenerated to be line segments.

For $\varepsilon = 10^{-15}$ and (at first "just") $k = 2$, we were running the algorithm more than $10^8$ times. During those experiments, we had fixed points $p = (15, 0)$ and $q = (120, 480)$ but randomly created $k$ disjoint line segments for each experiment. The maximum and mean values of $\kappa(\varepsilon)$ observed in those experiments are 77 170 and 4·75, respectively. We also recorded the length $L_{200}$ of the path obtained in the 200-th iteration if $\kappa(\varepsilon) > 200$ for those experiments. We assume that the final output path is a "very good" (for common application areas) approximation path because $\varepsilon = 10^{-15}$ is already very small. Then we obtained the approximation factor $L_{200}/L$ of our algorithm if we terminate the algorithm after 200 iterations, where $L$ is the length of the output "true" path. We observed that $L_{200}/L < 1{\cdot}19$ for all our inputs.

Then we also changed the value of $k$ to be a "small" integer between 3 and 20, and performed again a large number of experiments for studying the approximation factor $L_{200}/L$. Again we observed that $L_{200}/L < 1{\cdot}19$ for at least $10^5$ inputs for any of those $k = 3, \ldots, 20$.

The time complexity of the main algorithm in $\mathbb{R}^2$ (shown in Fig. 5) can be analysed as follows: Line 1 can be computed in $\mathcal{O}(mk)$ time. Line 2 can be computed in $\mathcal{O}(n + (k/\sqrt{\varepsilon_0}) \log(k/\varepsilon_0))$ time. Lines 3, 4, 7, 11 and 13 can be computed in $\mathcal{O}(k)$ time. Lines 5 and 9 can be computed in $\kappa(\varepsilon) \cdot \mathcal{O}(k)$ time. Line 6 and 10 requires $\mathcal{O}(k^2)$ time. Thus, we obtain the following

**Theorem 1.** *The main algorithm in $\mathbb{R}^2$ can be computed in*

$$\mathcal{O}(mk + (k/\sqrt{\varepsilon_0}) \log(k/\varepsilon_0) + k^3 + \kappa(\varepsilon) \cdot k)$$

*time to obtain an $L(\rho')/(L(\rho) - \varepsilon_0)$-approximation path between $p$ and $q$ among pairwise disjoint disks.*

We may adjust parameters $m$, $\varepsilon_0$, $\varepsilon$ to let the approximation factor $L(\rho')/(L(\rho) - \varepsilon_0)$ be sufficiently small and, at the same time, to terminate the while loop quickly.

The time complexity of the main algorithm in $\mathbb{R}^3$ (shown in Fig. 8) can be analysed as follows: Line 2 requires $\mathcal{O}(n + (k^4/{\varepsilon_0}^7) \log^3(k/\varepsilon_0))$ time. The other lines in this algorithm can be analysed exactly the same way as those of the main algorithm in $\mathbb{R}^2$ (shown in Figure 5).

Thus, we have the following

**Theorem 2.** *The main algorithm in $\mathbb{R}^3$ can be computed in*

$$\mathcal{O}(mk + (k^4/{\varepsilon_0}^7) \log^3(k/\varepsilon_0) + \kappa(\varepsilon) \cdot k)$$

*time to obtain an $L(\rho')/(L(\rho) - \varepsilon_0)$-approximation path between $p$ and $q$ among pairwise disjoint spheres.*

Again, we may adjust parameters $m$, $\varepsilon_0$, $\varepsilon$ for having the approximation factor $L(\rho')/(L(\rho) - \varepsilon_0)$ be sufficiently small and, at the same time, to terminate the while loop quickly.

## 5 Conclusion

We proposed an $L(\rho')/(L(\rho) - \varepsilon_0)$-approximation algorithm for computing an approximate Euclidean shortest path between two points passing through the free space $\mathcal{F}(\mathcal{S})$ containing a finite unordered set of pairwise disjoint spheres, where $L(\rho)$ and $L(\rho')$ are the lengths of the output path and the initial path, respectively, $\varepsilon_0$ is an accuracy parameter for the used Agarwal et al. algorithm. This algorithm provides an approximate answer to an open problem in computational geometry. The algorithm for spheres was presented by discussing at first the 2-dimensional case of disks, and then showing the analogy of a solution in 3D space.

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