

Watchman, Safari, and Zookeeper Routes in a Simple Polygon with Rubberband Algorithms

Fajie Li¹ and Reinhard Klette²

¹ College of Computer Science and Technology
Huaqiao University, Quanzhou, Fujian, China
Email: li.fajie@yahoo.com

² Computer Science Department, The University of Auckland
Private Bag 92019, Auckland 1142, New Zealand

Abstract. So far, the best result in running time for solving the “floating” watchman route problem (WRP) is $\mathcal{O}(n^4 \log n)$; the best result in running time for solving the “fixed” safari route problem (SRP) is $\mathcal{O}(n^2 \log n)$ published in 2003 by M. Dror, A. Efrat, A. Lubiw, and J. Mitchell. The best result in running time for solving the “floating” zookeeper route problem (ZRP) is $\mathcal{O}(n^2)$ published in 2001 by X. Tan. This paper provides an algorithm for the “floating” WRP with $\kappa(\varepsilon) \cdot \mathcal{O}(kn)$ runtime, where n is the number of vertices of the given simple polygon Π , and k the number of essential cuts; $\kappa(\varepsilon)$ defines the numerical accuracy in dependency of a selected constant $\varepsilon > 0$. This paper also provides an algorithm for the “floating” SRP with $\kappa(\varepsilon) \cdot \mathcal{O}(kn + m_k)$ runtime, where n is the number of vertices of the given simple polygon Π , k the number of convex polygon P_i s, and m_k is the number of vertices of P_i s. This paper also provides an algorithm for the “floating” ZRP with $\kappa(\varepsilon) \cdot \mathcal{O}(kn)$ runtime, where n is the number of vertices of all polygons involved, k the number of the “cages”.

Moreover, our algorithms are significantly simpler, easier to understand and implement than previous ones for solving the WRP, SRP and ZRP. Finally, our algorithms can solve more general SRP and ZRP where each convex polygon is replaced by a convex region such as convex polybezier (bezierngon) or ellipse.

1 Introduction

1.1 The Watchman Route Problem

Let Π be a planar, simple, topologically closed polygon with n vertices. A point $p \in \Pi$ is *visible* from point $q \in \Pi$ iff $pq \subset \Pi$; see Figure 1. The *watchman route problem* (WRP) of computational geometry, as discussed in [4], is defined as follows:

Calculate a shortest route $\rho \subset \Pi$ such that any point $p \in \Pi$ is visible from at least one point on ρ .

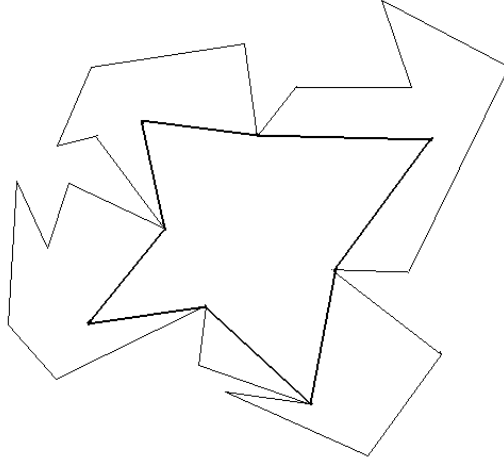


Fig. 1. A watchman route.

It follows that ρ is a polygonal path, and this problem is actually equivalent to the task that all points $p \in \Pi$ are visible just from the vertices of the polygonal path ρ . If the start point of the route is given then this refined problem is known as the *fixed* WRP.

A simplified WRP (i.e., find a shortest route in a simple isothetic polygon) was solved in 1988 in [17] by presenting an $\mathcal{O}(n \log \log n)$ algorithm. In 1991, [18] claimed to have presented an $\mathcal{O}(n^4)$ algorithm, solving the fixed WRP. In 1993, [48] obtained an $\mathcal{O}(n^3)$ solution for the fixed WRP. In the same year, this was further improved to a quadratic time algorithm [49]. However, four years later, in 1997, [26] pointed out that the algorithms in both [18] and [48] were flawed, but presented a solution for fixing those errors. Interestingly, two years later, in 1999, [51] found that the solution given by [26] was also flawed! By modifying the (flawed) algorithm presented in [48], [51] gave an $\mathcal{O}(n^4)$ runtime algorithm for the fixed WRP.

In 1995, [12] proposed an $\mathcal{O}(n^{12})$ runtime algorithm for the WRP. In the same year, [40] gave an $\mathcal{O}(n^6)$ algorithm for the WRP. This was improved in 2001 by an $\mathcal{O}(n^5)$ algorithm in [52]; this paper also proved the following

Theorem 1. *There is a unique watchman route in a simple polygon, except for those cases where there is an infinite number of different shortest routes, all of equal length.*

So far the best known result for the WRP is due to [22] which presented in 2003 an $\mathcal{O}(n^4 \log n)$ runtime algorithm.

Given the time complexity of those algorithms for solving the WRP, finding efficient approximation algorithms became an interesting subject. Recall the following definition; see, for example, [28]:

Definition 1. *An algorithm is an δ -approximation algorithm for a minimization problem P iff, for each input instance I of P , the algorithm delivers a solution that is at most δ times the optimum solution.*

In case of the WRP, the optimum solution is defined by the length of the shortest path. In 1995, [34] published an $\mathcal{O}(\log n)$ -approximation algorithm for solving the WRP. In 1997, [13] gave a 99-98-approximation algorithm with time complexity $\mathcal{O}(n \log n)$ for the WRP. In 2001, [54] presented a linear-time algorithm for an approximative solution of the fixed WRP such that the length of the calculated watchman route is at most twice of that of the shortest watchman route. The coefficient of accuracy was improved to $\sqrt{2}$ in [56] in 2004. Most recently, [57] presented a linear-time algorithm for the WRP for calculating an approximative watchman route of length at most twice of that of the shortest watchman route.

There are several generalizations and variations of watchman route problems; see, for example, [10, 11, 14, 21, 23–25, 31, 36, 39–43]. [1–3] show that some of these problems are NP-hard, and the authors solve them by approximation algorithms.

Let ESP denote the class of any Euclidean shortest path problem. Corresponding to Definition 1, we introduce the following

Definition 2. *An Euclidean path is an δ -approximation (Euclidean) path for an ESP problem iff its length is at most δ times the optimum solution.*

1.2 The Safari Route Problem

Let Π contain k pairwise disjoint convex polygons P_i ($i = 0, 1, \dots, k-1$) such that exactly one edge of each of these polygons P_i is incident with the boundary of Π . We are interested in a route with vertices $p_i \in \partial P_i$, for $i = 1, 2, \dots, k$. We consider indices modulo k , and identify index k with index 0 this way. Start point p and end point q are assumed to be identical; that means we have $p = q = p_0 \in \partial P_0$. Let P_i° be the interior of P_i . The *safari* route problem (SRP), introduced in [42], is defined as follows:

Calculate a shortest route $\rho \subset \Pi$ inside of Π such that ρ visits each P_i at a point p_i on the boundary of P_i , allowing that straight segments $p_i p_{i+1} \pmod{k}$ intersect P_i° in one or several segments, for $i = 0, 1, \dots, k-1$.

Analogous to WRP, if the start point of the route is given then this refined safari problem is known as the *fixed* SRP. Otherwise, the problem is known as the *floating* SRP.

[42] was historically (in 1992) the first publication which studied the safari problem. It claimed to have an $\mathcal{O}(kn^2)$ time algorithm for solving this problem, where n is the total number of vertices of polygon Π and all polygons P_i , for $i = 1, 2, \dots, k$. In 1994, [50] improved the result to an $\mathcal{O}(n^2)$ time algorithm for the floating SRP, not using anymore the restriction used in [42] of forcing the

route through a specific point. In 2003, [55] showed that there is an error in the algorithm proposed in [42], and presented an $\mathcal{O}(n^3)$ time algorithm for the fixed SRP, where n is again the total number of vertices of Π and all polygons P_i , for $i = 1, 2, \dots, k$. The algorithm runs in $\mathcal{O}(n^4)$ time for the floating SRP. In the same year, the result was improved by [22] with an algorithm running in time $\mathcal{O}(kn \log(n/k))$ for the fixed SRP.

This paper provides an algorithm for the floating SRP with $\kappa(\varepsilon) \cdot \mathcal{O}(kn + m_k)$ runtime, where n is the number of vertices of the given simple polygon Π , k the number of convex polygon P_i s, and m_k is the number of vertices of P_i s; $\kappa(\varepsilon)$ defines the numerical accuracy in dependency of a selected constant $\varepsilon > 0$.

1.3 The Zookeeper Route Problem

The *zookeeper* route problem (ZRP) was introduced in [19]; it is defined as follows (informally speaking, “the zookeeper is not supposed to enter any of the cages on his path, but to visit all”):

Calculate a shortest route ρ inside of Π such that ρ visits each “cage” P_i (i.e., convex polygons) at a point p_i on its boundary, and such that this path is not intersecting any of the interiors P_i° , for $i = 0, 1, \dots, k - 1$.

Analogous to WRP and SRP, if the start point of the route is given then this refined zookeeper route problem is known as the *fixed* ZRP. Otherwise, the problem is known as the *floating* ZRP.

Both [16] (in 1987) and [19] (in 1992) present an $\mathcal{O}(n^2)$ algorithm for the fixed ZRP, where n is the total number of vertices of Π and P_i , and $i = 1, 2, \dots, k$. In 1994, [27] improved this to $\mathcal{O}(n \log^2 n)$. In 2003, [5] improved this further to $\mathcal{O}(n \log n)$; the algorithm is still for the fixed ZRP, and n is the input size as defined above. [53] (in 2001) gave an $\mathcal{O}(n^2)$ algorithm for the floating ZRP.

This paper provides an algorithm for the floating ZRP with $\kappa(\varepsilon) \cdot \mathcal{O}(kn)$ runtime, where n is the number of vertices of the given simple polygon Π and all convex polygon P_i s, k the number of convex polygons P_i , $\kappa(\varepsilon)$ defines as in previous subsection.

This paper is organized as follows: Section 2 defines some notations for later usage. Section 3 briefly introduces the original rubberband algorithm to the computational geometry community. Section 4 presents a simple rubberband algorithm. Sections 5–7 propose and discuss the main algorithms of this paper. Section 8 handles more general SRP and ZRP. Section 9 concludes.

2 Basics

Throughout the paper, Π is a planar, simple, topologically closed polygon, and $\partial\Pi$ is its frontier, given by a polygonal path with n vertices. A vertex v of Π is called *reflex* if v ’s internal angle is greater than 180° .

Let P be a simple polygon. Let $q_1, q_2 \in \partial P$ with $q_1 \neq q_2$. We denote by $\rho(q_1, P, q_2)$ [$\rho^{-1}(q_1, P, q_2)$] a simple polyline which starts at q_1 and goes then anti-clockwise [clockwise] around ∂P ; finally it ends at q_2 . (Note that “anti-clockwise” means that when tracing an edge of ∂P , a point $p \in P^\circ$, which is sufficiently close to this edge, is always on the left-hand side of this edge.)

We recall some definitions from [57]. Let u be a vertex of Π which is adjacent to a reflex vertex v . Assume that the straight line uv intersects an edge of Π at v' . Then the segment $C = vv'$ partitions Π into two parts. C is called a *cut* of Π , and v is called a *defining vertex* of C . That part of Π which does not contain u is called *essential* part of C and denoted by $\Pi(C)$.

A cut C *dominates* a cut C' iff $\Pi(C)$ contains $\Pi(C')$. A cut is called *essential* if it is not dominated by another cut. In Figure 2 (which is Figure 1 in [57]), the cuts xx' and yy' are dominated by C_2 and C_5 , respectively; the cuts C_1, C_2, C_3, C_5 and C_4 are essential. Let \mathcal{C} be the set of all essential cuts. The WRP is reduced to find a shortest route ρ such that ρ visits all cuts in \mathcal{C} in some order (see Lemma 1, and also see [2] or [11]).

Let $S_C = \{C_1, C_2, \dots, C_k\}$ be the sorted set \mathcal{C} such that v_i is a defining vertex of C_i , and vertices v_i are distributed in clockwise order around $\partial \Pi$, for $i = 1, 2, \dots, k$.

If two points u and v are on two different edges of Π , such that the segment uv partitions Π into two parts, then we say that uv is a *general cut* of Π . We may arbitrarily select one of both endpoints of the segment uv , say u , to be its *start point*. For example, for a cut C of Π , we may identify the defining vertex of C with its start point.

If C_0, C_1, \dots, C_{k-1} are k general cuts of Π such that their start points are ordered clockwise around $\partial \Pi$, then we say that C_0, C_1, \dots, C_{k-1} and Π *satisfy the condition of the general watchman route problem (WRP)*. For example, in Figure 3, C_1, C_2 and Π satisfy the condition of the general WRP.

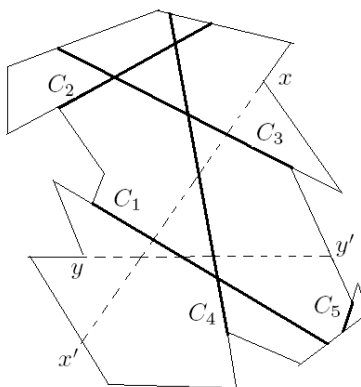


Fig. 2. Examples for cuts and essential cuts [57].

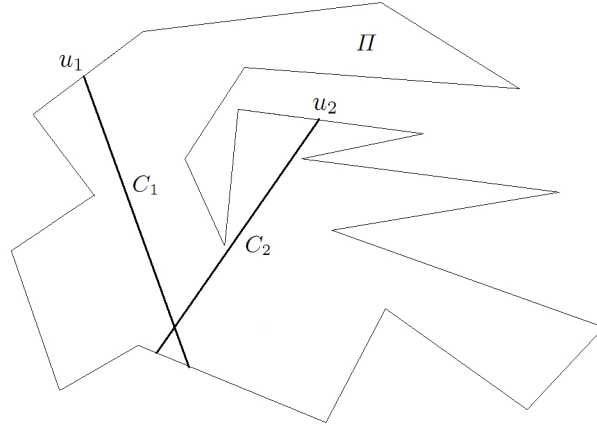


Fig. 3. Two general cuts C_1 , C_2 and a simple polygon Π which satisfy the condition of the general watchman route problem.

Let $p, q \in \Pi$; if $pq \subset \Pi$ then q can see p (with respect to Π), and p is a *visible point* of q .

Let $q \in \Pi$ and assume a segment $s \subset \Pi$. If, for each $p \in s$, q can see p , then we say that q can see s .

Let $q \in \Pi$, segment $s \subset \Pi$, $p \in s$, and p is not an endpoint of s . Let d_e be the Euclidean distance. If q can see p , but for any sufficiently small $\varepsilon > 0$, q cannot see p' , where $p' \in s$ and $d_e(p, p') = \varepsilon$, then we say that p is a *visible extreme point* of q (with respect to s and Π).

Let segment $s \subset \Pi$ and $q \in \Pi \setminus s$. If there exists a subsegment $s' \subseteq s$ such that q can see s' , and each endpoint of s' is a visible extreme point of q or an endpoint of s , then we say that s' is a *maximal visible segment* of q (with respect to Π).

Let s_0, s_1, \dots , and s_{k-1} be k segments ($k \geq 2$) in three-dimensional Euclidean space (in short: $3D$), $p \in s_0$, and $q \in s_{k-1}$. Let $L_S(p, q)$ be the length of the shortest path, starting at p , then visiting segments s_1, \dots , and s_{k-2} in order, and finally ending at q , where $S = \langle s_0, s_1, \dots, s_{k-1} \rangle$. Subscripts related to essential cuts are taken *mod* k in the following.

Let $p, q \in \Pi$. We denote by $\rho_\Pi(p, q)$ the shortest path from p to q inside of Π ; $L_\Pi(p, q)$ the length of $\rho_\Pi(p, q)$.

Let ρ be a polygonal path and $V(\rho)$ the set of all vertices of ρ ; $|V(\rho)|$ is the number of vertices of ρ . Denote by $C(S)$ the convex hull of a set S . Let S_0, S_1, \dots , and S_{k-1} be k non-empty sets; let

$$S_0 \times S_1 \times \dots \times S_{k-1} = \{(x_0, x_1, \dots, x_{k-1}) : x_0 \in S_0 \wedge x_1 \in S_1 \dots \wedge x_{k-1} \in S_{k-1}\}$$

Denoted by $\prod_{i=0}^{k-1} S_i$. If $S_i = [0, 1]$, where $i = 0, 1, 2, \dots, k-1$, then denote $\prod_{i=0}^{k-1} S_i$ by $[0, 1]^k$.

If an expression is derived from a finite number of polynomials in x by only applying operations “+”, “-”, “ \times ”, “ \div ”, or “ $\sqrt{}$ ” finitely often, then we say that this expression is a *simple compound of polynomials* in x . For example,

$$\sqrt{2x^2 + 1} - \frac{\sqrt{4x^5 - 1}}{\sqrt{3x^4 + 19}}$$

is a simple compound of polynomials in x .

Let f be a function, mapping \mathbb{R} into \mathbb{R} . If interval $J \subseteq I$, then we say that J is a *subinterval* of interval I . If f is monotonous in J , then we say that J is a *monotonous interval of f in I* . If x_0 satisfies $f(x_0) = 0$, and for a sufficiently small numbers $\delta > 0$ and all x_1 in the interval $(x_0 - \delta, x_0 + \delta)$, $f(x_1) \neq 0$, then we say that x_0 is an *isolated solution* of $f(x)$. If $I \subset \mathbb{R}$ is a bounded interval, and for all x in I , $f(x) = 0$, then we say that I is an *interval solution* to $f(x)$.

We generalize those two definition for the multi-variable case: Let f be a function from \mathbb{R}^m into \mathbb{R} , for $m \geq 1$. At a point $(x_{1_0}, x_{2_0}, \dots, x_{m_0})$, assume that $f(x_{1_0}, x_{2_0}, \dots, x_{m_0}) = 0$, and for a sufficiently small number $\delta > 0$ and all $(x_{1_1}, x_{2_1}, \dots, x_{m_1})$ such that $i = 1, 2, \dots, m$, and

$$x_{i_1} \in (x_{i_0} - \delta, x_{i_0} + \delta) \setminus \{x_{i_0}\}$$

we have that $f(x_{1_1}, x_{2_1}, \dots, x_{m_1}) \neq 0$; then point $(x_{1_0}, x_{2_0}, \dots, x_{m_0})$ is an *isolated solution* of $f(x_1, x_2, \dots, x_m)$.

If $I_1 \subset \mathbb{R}$ is a bounded interval, and for $i = 2, 3, \dots, m$, $I_i \subset \mathbb{R}$ is a bounded interval or a single point (i.e., a degenerated bounded interval), and for all $x_{1_1} \in I_1$, there exists an x_{i_1} in I_i such that $f(x_{1_1}, x_{2_1}, \dots, x_{m_1}) = 0$, then we say that $\langle I_1, I_2, \dots, I_m \rangle$ is an *interval solution* to $f(x_1, x_2, \dots, x_m)$. We say that $(x_{1_0}, x_{2_0}, \dots, x_{m_0})$ is an *isolated solution* to the system formed by $f_j(x_1, x_2, \dots, x_m) = 0$, for $j = 1, 2, \dots, m$, if for any of those j , $(x_{1_0}, x_{2_0}, \dots, x_{m_0})$ is an isolated solution to $f_j(x_1, x_2, \dots, x_m)$.

We say that $\langle I_1, I_2, \dots, I_m \rangle$ is an *interval solution* to the system formed by $f_j(x_1, x_2, \dots, x_m) = 0$, for $j = 1, 2, \dots, m$, if for any of those j , $\langle I_1, I_2, \dots, I_m \rangle$ is an *interval solution* to $f_j(x_1, x_2, \dots, x_m)$.

Let P be a convex polygon such that exactly one edge of P , denoted by e , is incident with the boundary of Π . If P' is a convex polygon such that P' contains P ; P' is similar to P ; and the distance between each edge of P' and its corresponding edge in P is $\varepsilon_3 > 0$, then P' is called a ε_3 -*outer polygon* of P (see Figure 4 left). Let l_e be the straight line which contains e . l_e cuts P' into two polygons. One of them contains P . We call this polygon ε_3 -*polygon* of P (with respect to e)(see Figure 4 right).

If P_0, P_1, \dots, P_{k-1} , and Π are k convex polygons such that P_i and P_{i+1} are disjointed; P_i is contained in Π and exactly one edge e_i of P_i is incident with $\partial\Pi$, where $i = 0, 1, 2, \dots, k-1$; and e_0, e_1, \dots, e_{k-1} are ordered clockwise around $\partial\Pi$, then we say that P_0, P_1, \dots, P_{k-1} , and Π *satisfy the SRP (or ZRP) condition*. For example, in Figure 5, P_1, P_2 , and Π satisfy the SRP (ZRP) condition.

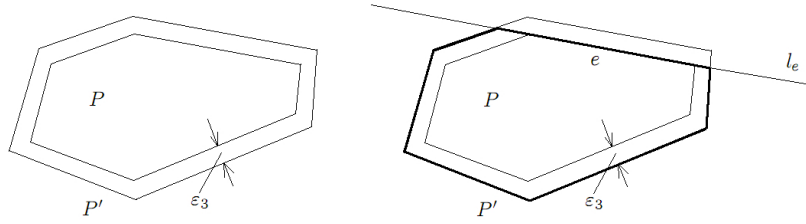


Fig. 4. Left: ε_3 -outer polygon of a convex polygon P ; right: ε_3 polygon of P .

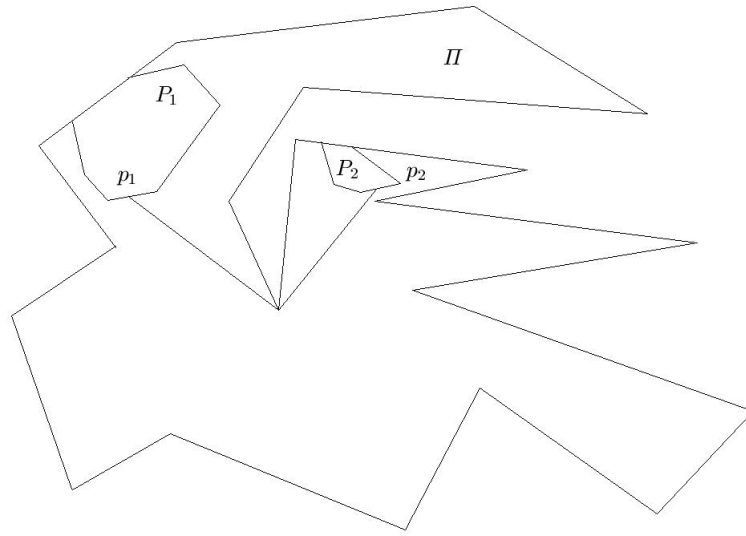


Fig. 5. An example for illustrating three polygons satisfying the safari problem condition.

Let P_i and Π satisfy the ZRP condition; $P_i \cap \Pi = e_i$ (i.e., e_i is an edge of P_i which is incident with $\partial\Pi$). If two points $p_{i_1}, p_{i_2} \in \partial P_i \setminus e_i$, and p_{i_1}, p_{i_2} (p_{i_1} and p_{i_2} may be identical) are ordered anti-clockwise around ∂P_i , then we say that (p_{i_1}, p_{i_2}) is a *ZRP pair*.

Let Π' be the simple polygon obtained by modifying Π by replacing e_i by $\partial P_i \setminus e_i$, where $i = 0, 1, 2, \dots, k-1$.

A *simple curve* is a curve in a plane such that each point of it can be expressed as $(x(t), y(t))$, where t varies in an interval $[a, b]$ and both $x(t)$ and $y(t)$ are differentiable in $[a, b]$. A *simple region* is a planar, convex, topologically closed region such that its boundary consists of some simple curves.

This ends our introduction of technical terms. We also recall in one place here four results and three algorithms which will be cited later in this paper:

Lemma 1. ([17], Lemma 3.3) *A solution to the watchman route problem (i.e., a shortest tour) must visit the C_i s in the same order as they meet $\partial\Pi$.*

Lemma 2. ([42], Lemma 2) *A solution to the safari problem (shortest tour) must visit the P_i 's in the same order as it meets $\partial\Pi$.*

Lemma 3. ([19], Lemma 2) *A solution to the zookeeper problem (shortest tour) must visit the P_i 's in the same order as it meets $\partial\Pi$.*

Theorem 2. ([57], Theorem 1) *Given a simple polygon Π ; the set \mathcal{C} of all essential cuts for the watchman route in Π can be computed in $\mathcal{O}(n)$ time.*

Algorithm 2D ESP (see [37], pages 639–641) has as input a simple, topologically closed polygon Π and two points $p, q \in \Pi$ and calculates a set of vertices of a shortest path from p to q inside of Π . Its running time is $\mathcal{O}(|V(\partial\Pi)|)$.

We also refer to a *convex hull algorithm* (see, e.g., [35] or Figure 13.7, [30]), which reads an ordered sequence of vertices of a planar simple polygonal curve ρ , and outputs an ordered sequence of vertices of the convex hull of ρ ; its running time is $\mathcal{O}(|V(\rho)|)$.

We will also use a *tangent calculation* (see [47]), which has as input an ordered sequence of vertices of a planar convex polygonal curve $\partial\Pi$ and a point $p \notin \Pi$; its output are two points $t_i \notin \Pi$ such that pt_i are tangents to $\partial\Pi$, for $i = 1, 2$; the running time is $\mathcal{O}(\log |V(\partial\Pi)|)$.

3 The Origin of Rubberband Algorithms

In this section, we recall the original rubberband algorithm, as defined for regular grids in 3D [9]. The ideas and basic steps of this algorithm will then guide us when solving the WRP. The *original* (or *first*) rubberband algorithm was defined in the context of 3D digital geometry, assuming a regular orthogonal grid in 3D.

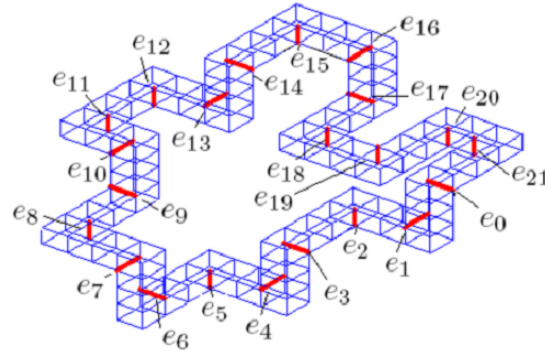


Fig. 6. Example of a cube-curve which has 22 critical edges.

A *cube-curve* g is a loop of face-connected grid cubes in a 3D regular orthogonal grid; the union \mathbf{g} of those cubes defines the *tube* of g . The original rubberband algorithm in [9] discusses ESPs in such tubes, which are also called *minimum-length polygonal curves* (MLPs). A *critical edge* of a cube-curve g is such a grid edge which is incident with exactly three different cubes contained in g . Figure 6 shows all the critical edges of a cube-curve.

The computation of 3D MLPs was at first published in [7–9, 29], proposing and discussing a *rubberband algorithm*¹. This *original rubberband algorithm* is also published in the book [30].

Let $\rho = (p_0, p_1, \dots, p_m)$ be a polygonal curve contained in a tube \mathbf{g} . Such a curve is *complete* if it intersects with every cube of g . A polygonal curve ξ is a *g-transform* of ρ iff ξ may be obtained from ρ by a finite number of steps, where each step is a replacement of a triple a, b, c of vertices by a polygonal sequence a, b_1, \dots, b_k, c such that the polygonal sequence a, b_1, \dots, b_k, c is contained in the same set of cubes of g as the polygonal sequence a, b, c .

Assume a polygonal curve $\rho = (p_0, p_1, \dots, p_m)$ and three pointers addressing vertices at positions $i-1, i$ and $i+1$ in this curve. There are three different *options* that may occur, and which define a specific *g-transform*:

(O_1) Point p_i can be deleted iff $p_{i-1}p_{i+1}$ is a line segment within the tube. Then the subsequence (p_{i-1}, p_i, p_{i+1}) is replaced in the curve by (p_{i-1}, p_{i+1}) . In this case, the algorithm continues with vertices $p_{i-1}, p_{i+1}, p_{i+2}$.

(O_2) The closed triangular region $\triangle(p_{i-1}, p_i, p_{i+1})$ intersects more than just three critical edges of cube-curve g (i.e., a simple deletion of p_i would not be sufficient anymore). This situation is solved by calculating a convex arc and by replacing point p_i by a sequence of vertices q_1, \dots, q_k on this convex arc between p_{i-1} and p_{i+1} such that the sequence of line segments $p_{i-1}q_1, \dots, q_k p_{i+1}$ lies within the tube. In this case, the algorithm continues with a triple of vertices starting with the calculated new vertex q_k .

If (O_1) and (O_2) do not lead to any change, the third option may lead to an improvement (i.e., a shorter polygonal curve which is still contained and complete in the given tube). Here, l_e denotes the straight line defined by extending an edge e at both ends to infinity:

(O_3) Point p_i may be moved on its critical edge to obtain an optimum position p_{new} minimizing the total length of both line segments $p_{i-1}p_{new}$ and $p_{new}p_{i+1}$. First, find $p_{opt} \in l_e$ such that

$$|p_{opt} - p_{i-1}| + |p_{opt} - p_{i+1}| = \min_{p \in l_e} L(p)$$

with $L(p) = |p - p_{i-1}| + |p - p_{i+1}|$. Then, if p_{opt} lies on the closed critical edge e , let $p_{new} = p_{opt}$. Otherwise, let p_{new} be that vertex bounding e and lying closest to p_{opt} .

The authors showed in various previous publications (see, for example, [32]) that the basic idea of (O_3) can be generalized to establish a whole class of rubberband algorithms (RBAs) for solving various Euclidean shortest path problems.

¹ Not to be confused with a 2D image segmentation algorithm of the same name [33].

The main algorithm of this paper is also just some kind of adaptation of this original rubberband algorithm.

4 Sequences of Line Segments in 3D

In this section, we present a simple rubberband algorithm which receives as input a finite sequence of line segments in 3D. Later it will be generalized and becomes then the main algorithm; see Section 5.

4.1 The Algorithm

The numerical accuracy of results obtained by a rubberband algorithm is controlled by a chosen accuracy constant $\varepsilon > 0$. For example, with respect to current computer technology, a constant such as $\varepsilon = 10^{-15}$ is appropriate. However, whenever ε is mentioned, have in mind that its value may further decrease with the progress in computer technology.

Algorithm 1 (RBA for a sequence of pairwise disjoint 3D line segments)

Input: A sequence of k pairwise disjoint line segments S_1, S_2, \dots, S_k in 3D; two points $p, q \notin \bigcup_{i=1}^k S_i$, and an accuracy constant $\varepsilon > 0$.

Output: A sequence $\langle p, p_1, p_2, \dots, p_k, q \rangle$ of an $[1 + 4(k+1)r(\varepsilon)/L]$ -approximation path which starts at p , then visits (i.e., passes through) segments S_i at p_i in the given order, and finally ends at q , where L is the length of an optimal path, $r(\varepsilon)$ the upper error bound² for distances between p_i and the corresponding optimal vertex p'_i : $d_e(p_i, p'_i) \leq r(\varepsilon)$, for $i = 1, \dots, k$, where d_e denotes the Euclidean distance.

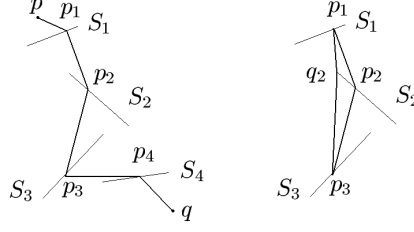
We provide an informal specification of the algorithm. The algorithm consists of two parts: initialization and iteration step. In the initialization part, we select an initial path and calculate its length. For example, take arbitrarily one point in each segment and connect those points into a sequence, to obtain an initial path. We could also take the center or one of the endpoints in each segment. In each iteration cycle, we update all the vertices of the path in sequence: For every three subsequent vertices p_{i-1} , p_i and p_{i+1} in the path, we consider the first and third vertices, p_{i-1} and p_{i+1} , as being fixed and slide p_i freely in segment S_i into an optimal point with respect to this local configuration; we update p_i by replacing it with this newly detected, locally optimal point. That is, we apply

$$d_e(p_{i-1}, p_i) + d_e(p_i, p_{i+1}) = \min\{d_e(p_{i-1}, p) + d_e(p, p_{i+1}) : p \in S_i\} \quad (1)$$

At the end of the iteration cycle we compute the difference between the length of the previous path to that of the current (i.e., updated) path; if this is less than ε then this terminates the algorithm. Otherwise, we go to the next iteration cycle.

Figure 7 shows on the left an initial path $\langle p, p_1, p_2, p_3, p_4, q \rangle$ for Algorithm 1; on the right it shows an optimal point $q_2 \in S_2$ for the given positions of p_1 and p_3 , defining the new position of p_2 . Note that optimal points are not necessarily at endpoint positions.

² It is obvious to see that $\lim_{\varepsilon \rightarrow 0} r(\varepsilon) = 0$

**Fig. 7.** Illustration for Algorithm 1.

4.2 Proof of Correctness

We have to show that these repeated local optimizations of Algorithm 1 ensure that the calculated path converges against the shortest path.

Let s_0, s_1 and s_2 be three (closed) pairwise disjoint line segments; the two endpoints of s_i be $a_i = (a_{i1}, a_{i2}, a_{i3})$ and $b_i = (b_{i1}, b_{i2}, b_{i3})$. A point $p_i \in s_i$, for $i = 0, 1, 2$, may be written as

$$\begin{aligned} p_i(t_i) &= a_i + (b_i - a_i)t_i \\ &= (a_{i1} + (b_{i1} - a_{i1})t_i, a_{i2} + (b_{i2} - a_{i2})t_i, a_{i3} + (b_{i3} - a_{i3})t_i) \end{aligned}$$

where $t_i \in [0, 1]$. Let $d(t_0, t_1, t_2) = d_e(p_1(t_1), p_0(t_0)) + d_e(p_1(t_1), p_2(t_2))$. Then we have the following

Observation 1

$$\frac{\partial d(t_0, t_1, t_2)}{\partial t_1} = 0 \quad (2)$$

implies that t_2 is a simple compound of polynomials of t_0 and t_1 . All the t_0, t_1 and t_2 are in $[0, 1]$.

Proof. The formula

$$d_e(p_1(t_1), p_0(t_0)) = \sqrt{\sum_{i=1}^3 \{[a_{1i} + (b_{1i} - a_{1i})t_1] - [a_{0i} + (b_{0i} - a_{0i})t_0]\}^2} \quad (3)$$

can be simplified: Without loss of generality, we can assume that s_1 is parallel to one of the three coordinate axes. It follows that only one element of the set $\{b_{1i} - a_{1i} : i = 1, 2, 3\}$ is not equal to 0, and the other two are equal to 0. Thus, we can assume that the expression inside the square root in Equation (3) can be written as

$$\begin{aligned} &\sum_{i=1}^3 \{[a_{1i} + (b_{1i} - a_{1i})t_1] - [a_{0i} + (b_{0i} - a_{0i})t_0]\}^2 \\ &= \{[a_{11} + (b_{11} - a_{11})t_1] - [a_{01} + (b_{01} - a_{01})t_0]\}^2 \\ &\quad + \{a_{12} - [a_{02} + (b_{02} - a_{02})t_0]\}^2 \\ &\quad + \{a_{13} - [a_{03} + (b_{03} - a_{03})t_0]\}^2 \end{aligned}$$

Thus, we have that

$$d_e(p_1, p_0) = |A_1| \sqrt{(t_1 + B_0 t_0 + C_0)^2 + D_0 t_0^2 + E_0 t_0 + F_0}$$

where A_1 is a function of a_{1_i} and b_{1_i} ; B_0, C_0, D_0, E_0 and F_0 are functions of $a_{0_i}, b_{0_i}, a_{1_i}$ and b_{1_i} , for $i = 0, 1, 2$. – Analogously, we have that

$$d_e(p_1, p_2) = |A_1| \sqrt{(t_1 + B_2 t_2 + C_2)^2 + D_2 t_2^2 + E_2 t_2 + F_2}$$

where B_2, C_2, D_2, E_2 and F_2 are functions of $a_{1_i}, b_{1_i}, a_{2_i}$ and b_{2_i} for $i = 0, 1, 2$. By Equation (2) or the following,

$$\frac{\partial(d_e(p_1, p_0) + d_e(p_1, p_2))}{\partial t_1} = 0$$

we have that

$$\begin{aligned} & \frac{t_1 + B_0 t_0 + C_0}{\sqrt{(t_1 + B_0 t_0 + C_0)^2 + D_0 t_0^2 + E_0 t_0 + F_0}} \\ & + \frac{t_1 + B_2 t_2 + C_2}{\sqrt{(t_1 + B_2 t_2 + C_2)^2 + D_2 t_2^2 + E_2 t_2 + F_2}} = 0 \end{aligned}$$

This equation can be written as

$$A t_2^2 + B t_2 + C = 0$$

where A, B , and C are polynomials of t_0, t_1 (and $a_{0_i}, b_{0_i}, a_{1_i}, b_{1_i}, a_{2_i}$ and b_{2_i} for $i = 0, 1, 2$). To keep t_2 inside of $[0, 1]$, let $t_2 = 0$ if we have to satisfy $t_2 < 0$; and let $t_2 = 1$ if we have to satisfy $t_2 > 1$. This proves the observation. \square

Analogously, we have

Observation 2 Equation (2) uniquely implies that t_1 is a continuous function in t_0 and t_2 .

Proof. We may translate two points $p_0(t_0)$ and $p_2(t_2)$, and line segment s_1 such that the endpoint a_1 of s_1 is identical to the origin. Then rotate $p_0(t_0), p_2(t_2)$, and s_1 such that the other endpoint b_1 of s_1 is (also) on the x -axis. Let $p_0(t_0) = (p_{0_1}, p_{0_2}, p_{0_3}), p_2(t_2) = (p_{2_1}, p_{2_2}, p_{2_3})$. After translation and rotation, we have that $a_1 = (0, 0, 0)$ and $b_1 = (b_{1_1}, 0, 0)$. Thus, $p_1(t_1) = (b_{1_1} t_1, 0, 0)$, and

$$d_e(p_1, p_0) = \sqrt{(b_{1_1} t_1 - p_{0_1})^2 + p_{0_2}^2 + p_{0_3}^2}$$

$$d_e(p_1, p_2) = \sqrt{(b_{1_1} t_1 - p_{2_1})^2 + p_{2_2}^2 + p_{2_3}^2}$$

Equation (2) is equivalent to

$$\frac{\partial(d_e(p_1, p_0) + d_e(p_1, p_2))}{\partial t_1} = 0$$

From this we obtain that

$$\frac{b_{1_1}t_1 - p_{0_1}}{\sqrt{(b_{1_1}t_1 - p_{0_1})^2 + p_{0_2}^2 + p_{0_3}^2}} + \frac{b_{1_1}t_1 - p_{2_1}}{\sqrt{(b_{1_1}t_1 - p_{2_1})^2 + p_{2_2}^2 + p_{2_3}^2}} = 0$$

This equation has a unique solution

$$t_1 = \frac{p_{0_1}\sqrt{p_{2_2}^2 + p_{2_3}^2} + p_{2_1}\sqrt{p_{0_2}^2 + p_{0_3}^2}}{b_{1_1}(\sqrt{p_{2_2}^2 + p_{2_3}^2} + \sqrt{p_{0_2}^2 + p_{0_3}^2})}$$

Again, to keep t_2 inside of $[0, 1]$, let $t_2 = 0$ if we have to satisfy $t_2 < 0$; and let $t_2 = 1$ if we have to satisfy $t_2 > 1$. This proves the observation. \square

Let s_0, s_1, \dots , and s_{k+1} be $k+2$ (closed) line segments. Let the two endpoints of s_i be $a_i = (a_{i_1}, a_{i_2}, a_{i_3})$ and $b_i = (b_{i_1}, b_{i_2}, b_{i_3})$. Points $p_i \in s_i$, for $i = 0, 1, 2, \dots, k+1$, can be written as follows:

$$\begin{aligned} p_i(t_i) &= a_i + (b_i - a_i)t_i \\ &= (a_{1_1} + (b_{1_1} - a_{1_1})t_i, a_{1_2} + (b_{1_2} - a_{1_2})t_i, a_{1_3} + (b_{1_3} - a_{1_3})t_i) \end{aligned}$$

where $t_i \in [0, 1]$.

Let

$$d(t_0, t_1, t_2, \dots, t_{k+1}) = \sum_{i=0}^k d_e(p_i(t_i), p_{i+1}(t_{i+1})) \quad (4)$$

Assume that both s_0 and s_{k+1} degenerate into single points p and q . Then we have that $t_0 = t_{k+1} = 0$. We also have the following

Observation 3 For each $i \in \{1, 2, \dots, k\}$,

$$\frac{\partial d(t_0, t_1, t_2, \dots, t_k, t_{k+1})}{\partial t_i} = 0 \quad (5)$$

is equivalent to

$$\frac{\partial d(t_{i-1}, t_i, t_{i+1})}{\partial t_i} = 0 \quad (6)$$

where t_1, t_2, \dots, t_k are in $[0, 1]$.

Note that Equation (5) is related to a global minimum property of the Euclidean path $\langle p, p_1, p_2, \dots, p_k, q \rangle$ while Equation (6) is related to a local minimum property of the same path. Therefore, Observation 3 describes a relationship between global and local minimum properties of the same path.

Observation 4 The equational system formed by Equation (5) (where $i = 1, 2, \dots, k$) implies a unary equation $f(t_1) = 0$ which has only a finite number of isolated or interval solutions in $[0, 1]$.

Proof. By Observation 3 and Observation 1, t_{i+1} is a simple compound of polynomials in t_{i-1} and t_i , denoted by $t_{i+1} = f_i(t_{i-1}, t_i)$. Thus, the system formed by Equation (5) (where $i = 1, 2, \dots, k$) implies an equational system formed by $t_2 = f_2(t_0, t_1)$, $t_3 = f_3(t_1, t_2)$, $t_4 = f_4(t_2, t_3)$, \dots , $t_k = f_k(t_{k-2}, t_{k-1})$, and $t_{k+1} = f_{k+1}(t_{k-1}, t_k)$. Now note that $t_0 = t_{k+1} = 0$. Therefore, $f(t_1)$ is a simple compound of polynomials in t_1 . Note that function $f(t_1)$ has only a finite number of monotonous intervals in $[0, 1]$, and $f(t_1)$ is differentiable in each of those monotonous intervals. Thus, $f(t_1)$ can be approximately expressed as a linear function in a finite number of monotonous subintervals in $[0, 1]$. Therefore, Function $f(t_1)$ has only a finite number of isolated or interval solutions in $[0, 1]$. This proves the observation. \square

Observation 5 *Algorithm 1 defines a continuous function $f_{RBA}(p_1, p_2, \dots, p_k)$ in $\prod_{i=1}^k S_i = S_1 \times S_2 \times \dots \times S_k$, or a function $f_{RBA}(t_1, t_2, \dots, t_k)$ in $\prod_{i=1}^k I_i = I_1 \times I_2 \times \dots \times I_k = [0, 1]^k$. And $f_{RBA}(t_1, t_2, \dots, t_k)$ has only a finite number of values.*

Proof. For each $(p_1, p_2, \dots, p_k) \in \prod_{i=1}^k S_i$ or each $(t_1, t_2, \dots, t_k) \in [0, 1]^k$, Algorithm 1 outputs the vertices of an approximate path. It can also output the length of the approximate path, which is a positive real. In this way, Algorithm 1 defines a mapping from $\prod_{i=1}^k S_i$ to \mathbb{R} , or from $[0, 1]^k$ into \mathbb{R} . By Observation 2, and because Algorithm 1 will terminate after a finite number of steps, thus, $f_{RBA}(t_1, t_2, \dots, t_k)$ is continuous in its domain $[0, 1]^k$.

To prove the second conclusion of the observation, it is sufficient to prove that for each interval solution J to the equational system formed by Equation (5) (where $i = 1, 2, \dots, k$), the following function

$$f_{RBA}(t_1, t_2, \dots, t_k) : J \rightarrow \mathbb{R}$$

has only a finite number of values. Suppose that $f(t_1) \equiv 0$, where t_1 is in an interval $I \subseteq [0, 1]$, and $f(t_1)$ is defined as in Observation 4. By Observation 4, $d(t_1, t_2, \dots, t_k)$ implies a unary length function $L(t_1) = 0$, where t_1 is in an interval $I' \subseteq I$, $d(t_1, t_2, \dots, t_k)$ is defined as in Equation (4), and

$$\frac{d[L(t_1)]}{dt_1} \equiv 0$$

($t_1 \in I' \subseteq I$). This implies that the length function $L(t_1) \equiv \text{constant}$, where $t_1 \in I' \subseteq I$. Thus, function

$$f_{RBA}(t_1, t_2, \dots, t_k) : J \rightarrow \mathbb{R}$$

has only a finite number of values. We have proven the observation. \square

Theorem 3. *If the chosen accuracy constant ε is sufficiently small, then, for any initial path, Algorithm 1 outputs a unique $[1 + 4(k+1) \cdot r(\varepsilon)/L]$ -approximation path.*

Proof. By Observation 5, Algorithm 1 defines a function $f_{RBA}(p_1, p_2, \dots, p_k)$ in $\prod_{i=1}^k S_i$ which is continuous and only maps into a finite number of positive real numbers (i.e., the lengths of paths), for any points p_1, p_2, \dots, p_k sampled in $\prod_{i=1}^k S_i$. Therefore, $f_{RBA}(p_1, p_2, \dots, p_k)$ must be a singleton.

For each $i \in \{1, 2, \dots, k-1\}$, the error of the difference between $d_e(p_i, p_{i+1})$ and $d_e(v_i, v_{i+1})$ is at most $4 \cdot r(\varepsilon)$ because of $d_e(p_i, v_i) \leq r(\varepsilon)$. Let $p = p_0 = v_0$ and $q = p_{k+1} = v_{k+1}$. We obtain that

$$L \leq \sum_{i=0}^k d_e(p_i, p_{i+1}) \leq \sum_{i=0}^k [d_e(v_i, v_{i+1}) + 4r(\varepsilon)] = L + 4(k+1)r(\varepsilon) \quad (7)$$

Thus, the output path is an $[1 + 4(k+1) \cdot r(\varepsilon)/L]$ -approximation path. This proves the theorem. \square

Based on

Proposition 1. (see [20, 46, 59]) *The shortest path from p to q , which passes through the interior points of a sequence $\langle S_1, S_2, \dots, S_k \rangle$ of line segments in the given order, is unique.*

and also based on our experiments we conclude that the equational system formed by Equation (5) (where $i = 1, 2, \dots, k$) has only isolated solutions in $[0, 1]$.

We implemented ³ Algorithm 1, and were running the program several thousands of times. For each run we took a random configuration of line segments S_1, S_2, \dots, S_k .

<i>min iterations</i>	2039	2888	2133
<i>max iterations</i>	3513	3243	8441
<i>min run time</i>	44·11 s	62·922 s	47·188 s
<i>max run time</i>	77·657 s	70·094 s	187·672 s
<i>min initial length</i>	827430	822952	822905
<i>max initial length</i>	846928	841860	839848
<i>min final length</i>	516994·66273890162	513110·99723050051	512768·28438387887
<i>max final length</i>	516994·66273896693	513110·99723056785	512768·28457121132

Table 1. Three examples of experimental results, for three randomly generated sequences of 5,000 line segments in 3D space.

For example, Table 1 shows the results for three random configurations of 5,000 line segments. Each column summarizes results corresponding to one configuration. For each configuration of line segments, we ran Algorithm 1 fifty times with 50 random initial paths when starting the program. The table shows

³ The source code can be downloaded at www.mi.auckland.ac.nz/; follow link MI-tech Reports.

that for each configuration, although the lengths of initial paths are different, the lengths of final paths are approximately identical.

In all those thousands of runs it turned out that the equational system formed by Equation (5) (where $i = 1, 2, \dots, k$) has only isolated solutions in $[0, 1]$.

4.3 A Degenerate Case

In this section we study a degenerate case: when applying (O_3) of the original rubberband algorithm (see Section 3), assume that at least two vertices of the obtained updated polygonal path are identical. In this case, RBAs may not work properly. Unfortunately, this may actually occur sometimes when working with RBAs, and we show (one possible way) how to handle such degenerate cases.

Having an option for dealing with such degenerate cases, we may even remove “pairwise disjoint” from the input conditions for the 3D line segments in Algorithm 1. The more general algorithm is now as follows:

Algorithm 2 (RBA for a sequence of arbitrary 3D line segments)

Input: A sequence of k line segments S_1, S_2, \dots, S_k in 3D; two points p and q which are both not in $\bigcup_{i=1}^k S_i$, and an accuracy constant $\varepsilon > 0$.

Output: Exactly the same as for Algorithm 1.

However, before specifying this algorithm, we discuss at first three examples, which will help to understand the issue of degenerated cases, and will then be used to motivate this modified RBA.

Example 1. Let the input for Algorithm 2 be as follows (see also Figure 8):

$$S_1 = q_1q_2, S_2 = q_2q_3, q_1 = (0, 0), q_2 = (2, 4), q_3 = (3, 0), p = (1, 0), \text{ and } q = (2, 0).$$

To initialize, let p_1 and p_2 be the centers of S_1 and S_2 , respectively [i.e., $p_1 = (1, 2)$, and $p_2 = (2.5, 2)$]. We obtain that the length of the initialized polyline $\rho = \langle p, p_1, p_2, q \rangle$ is equal to 5.5616. Algorithm 2 finds the shortest path $\rho = \langle p, p'_1, p'_2, q \rangle$ where $p'_1 = (0.3646, 0.7291)$, $p'_2 = (2.8636, 0.5455)$ and the length of it is equal to 4.4944 (see Table 2, which lists resulting δ s for the number t of iterations).

Example 2. Now we modify Example 1 such that $p_1 = p_2 = q_2$; in this case, the output of Algorithm 2 will be false: the calculated path equals $\rho = \langle p, p'_1, p'_2, q \rangle$, where $p'_1 = q_2$ and $p'_2 = q_2$, and its length equals 8.1231.

We call a situation as in the previous example a *degenerate case* when applying a rubberband algorithm. In general, it is defined by the occurrence of at least two identical vertices of the updated polygonal path. Such a degenerate case causes Algorithm 1 to fail.

A degenerate case can be solved approximately: we will not allow that a case $p_2 = q_2$ is happening. To do so, we can remove sufficiently small endsegments from both segments S_1 and S_2 . The following example shows how to handle such a degenerate case.

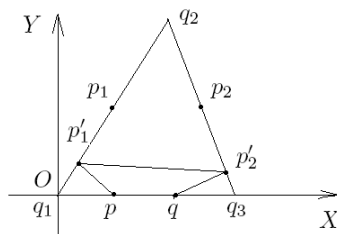


Fig. 8. Illustration of a degenerate case of a rubberband algorithm.

t	δ
1	-0.8900
2	-0.1752
3	-0.0019
4	-1.2935e-005
5	-8.4435e-008
6	-5.4930e-010
7	-3.5740e-012

Table 2. Number t of iterations and resulting δ s, for Example 1, illustrated by Figure 8, with $p_1 = (1, 2)$ and $p_2 = (2.5, 2)$ as initialization points.

Example 3. We modify the initialization step of Example 2 as follows: Let the accuracy be

$$\varepsilon = 1.0 \times 10^{-100}$$

and let

$$\begin{aligned} \delta' &= 2.221 \times 10^{-16} \\ x_1 &= 2 - \delta' & \text{and} & & y_1 &= 2 \times x_1 \\ x_2 &= 2 + \delta' & \text{and} & & y_2 &= -4 \times (x_2 - 3) \\ p_1 &= (x_1, y_1) & \text{and} & & p_2 &= (x_2, y_2) \end{aligned}$$

The length of the initialized polyline $\rho = \langle p, p_1, p_2, q \rangle$ is equal to 8.1231. Algorithm 2, to be defined below, will calculate the shortest path $\rho = \langle p, p'_1, p'_2, q \rangle$, where $p'_1 = (0.3646, 0.7291)$ and $p'_2 = (2.8636, 0.5455)$, and its length equals 4.4944 (see Table 3 for resulting δ_s in dependence of the number I of iterations).

Of course, if we leave the accuracy to be $\varepsilon = 1.0 \times 10^{-10}$, then the algorithm will stop sooner, after less iterations. For example, the algorithm was implemented on a Pentium 4 PC using Matlab 7.04. If we change the value of δ' into

$$\delta' = 2.22 \times 10^{-16}$$

then we obtain the same false result as that of Example 1. This is because this particular implementation was not able to recognize a difference between x_1 and $x_1 \mp 2.22 \times 10^{-16}$. However, for practical applications in general, the value

$$\delta' = 2.221 \times 10^{-16}$$

t	δ	t	δ	t	δ	t	δ
1	-5.4831e-007	7	-1.2313	13	-7.0319e-010	19	8.8818e-016
2	-6.2779e-006	8	-2.0286	14	-4.5732e-012	20	8.8818e-016
3	-7.7817e-005	9	-0.2104	15	-3.0198e-014	21	-8.8818e-016
4	-9.6471e-004	10	-0.0024	16	-8.8818e-016	22	8.8818e-016
5	-0.0119	11	-1.6550e-005	17	8.8818e-016	23	-8.8818e-016
6	-0.1430	12	-1.0809e-007	18	-8.8818e-016	24	0

Table 3. Number t of iterations and resulting δ s, for the example shown in Figure 8, with $p_1 = (2 - \delta', 2(2 - \delta'))$ and $p_2 = (2 + \delta', -4((2 + \delta') - 3))$ as initialization points and $\delta' = 2.221\text{e-}16$.

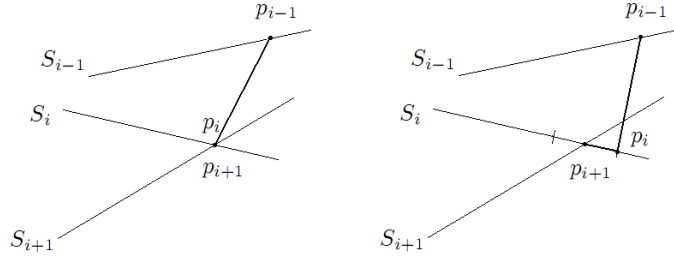


Fig. 9. Handling a degenerate case of a rubberband algorithm.

should be small or accurate enough for this implementation.

We summarize the method for handling a degenerate case with the modified rubberband algorithm (thus further preparing for defining Algorithm 2):

Let S_{i-1} , S_i and S_{i+1} be three continuous segments in the input such that $S_i \cap S_{i+1} \neq \emptyset$. Assume that p_{i-1} , p_i and p_{i+1} are three continuous vertices of the updated polygonal path such that p_i and p_{i+1} are identical (see left of Figure 9). Let ε_2 be a sufficiently small positive number. There are at most two possible points p in S_i such that $d_e(p, p_{i+1}) = \varepsilon_2$. Select one such point p such that $d_e(p, p_i) + d_e(p, p_{i+1})$ is smaller, and update the polygonal path by letting $p_i = p$ (see right of Figure 9). We say that p_i is ε_2 -transformed to be p in S_i . Analogously to the explanation of Equation (7), the total error of this ε_2 -transform is $4(k-1)\varepsilon_2$, for handling the degenerate case, and ε_2 is called a chosen *degeneration accuracy constant*.

To finalize this section, we provide a pseudo code of Algorithm 2 which also handles degenerate cases as discussed above. Let $p_0 = p$ and $p_{k+1} = q$. The output of this algorithm is a sequence $\langle p, p_1, p_2, \dots, p_k, q \rangle$ of an $\{1 + 4[(k+1)r(\varepsilon) + (k-1)\varepsilon_2]/L\}$ -approximation path which starts at p , then visits segments S_i at p_i in the given order, and finally ends at q , where L and $r(\varepsilon)$ are defined as in Algorithm 1, and ε_2 is a chosen degenerative accuracy constant. The pseudo code is listed below:

- 1: For each $i \in \{1, 2, \dots, k\}$, let p_i be a point in S_i such that p_i if $S_i \cap S_{i\mp 1} \neq \emptyset$, then select p_i such that p_i is not the intersection point.
- 2: Calculate $L_1 = \sum_{i=0}^{k-1} L_S(p_i, p_{i+1})$; and let $L_0 = 0$.
- 3: **while** $L_1 - L_0 > \varepsilon$ **do**
- 4: **for** each $i \in \{1, 2, \dots, k\}$ **do**
- 5: Compute a point $q_i \in S_i$ such that
 $d_e(p_{i-1}, q_i) + d_e(q_i, p_{i+1}) = \min\{d_e(p_{i-1}, p) + d_e(p, p_{i+1}) : p \in S_i\}$
- 6: **if** $S_i \cap S_{i\mp 1} \neq \emptyset$ and q_i is the intersection point **then**
- 7: ε_2 -transform q_i to be another point (still denoted by q_i) in S_i .
- 8: **end if**
- 9: Update the path $\langle p, p_1, p_2, \dots, p_k, q \rangle$ by replacing p_i by q_i .
- 10: **end for**
- 11: Let $L_0 = L_1$ and calculate $L_1 = \sum_{i=0}^{k-1} L_S(p_i, p_{i+1})$.
- 12: **end while**
- 13: Return $\{p, p_1, p_2, \dots, p_k, q\}$.

In other words, Algorithm 2 is modified from Algorithm 1 by adding Steps 6–8 in this pseudo code for handling the degenerate case. An informal specification of Algorithm 2 can also be obtained by modifying the informal specification of Algorithm 1 in Section 4.1 as follows:

For each updated point p_i in Equation (1), if it is the intersection point between S_i and S_{i-1} or S_{i+1} , then ε_2 -transform p_i into another point.

We call

$$\{S_1, S_2, \dots, S_k\}$$

the *step set* of the rubberband algorithm, and each S_i is a *step element* of the rubberband algorithm, where $i = 1, 2, \dots, k$.

4.4 Time Complexity

The time complexity of Algorithm 2 (see pseudo code at the end of Section 4.3) can be analyzed as follows: The main computation occurs in two stacked loops. Each iteration of the inner for-loop runs in time $\mathcal{O}(k)$. In theory, the outer while-loop might take

$$\kappa(\varepsilon) = \frac{L_0 - L}{\varepsilon}$$

times, where L is the length of an optimal path, L_0 is the length of an initial path. Thus, Algorithm 2 will run in time $\kappa(\varepsilon)\mathcal{O}(k)$.

Algorithm 1 has been implemented and tested for a large number of various inputs. We let the chosen accuracy constant to be $\varepsilon = 10^{-15}$, and generated input for $k = 5000$, $k=10000$, or $k=20000$.

For each of those different numbers of segments, we were running the program several thousands of times. According to the resulting observations, the program often terminates after k iterations. These cases occurred at more than 90% of all inputs. So far, the worst case for all the tested inputs was $3 \cdot 6k$ iterations, and worst cases in this order occurred at less than 0.1% of all inputs.

Based on these thousands of runs, we conclude that Algorithm 2 runs practically with $\kappa(\varepsilon) = \mathcal{O}(k)$, or, equivalently, in time $\mathcal{O}(k^2)$.

<i>min iterations</i>	2605	1522	2926
<i>max iterations</i>	3227	2741	7573
<i>min work time</i>	158.771sec	93.235sec	178.516sec
<i>max work time</i>	196.584sec	167.844sec	461.938sec
<i>min initial length</i>	3.32576e+006	3.33183e+006	3.33079e+006
<i>max initial length</i>	3.36785e+006	3.37652e+006	3.37889e+006
<i>min final length</i>	2085786.2964211311	2083340.4955095584	2068552.0753370232
<i>max final length</i>	2085786.2964214147	2083340.4955139237	2068552.0753745015

Table 4. Experiment results for three random configurations of 20000 line segments.

For example, Table 4 shows the results for three random configurations of 20000 line segments. Each column shows summary results corresponding to one configuration. For each configuration of line segments, we ran Algorithm 1 fifty times with 50 random initial paths for starting the program. The table shows that for each configuration, although the lengths of initial paths are different, the lengths of final paths are approximately identical.

5 Solving the Watchman Route Problem

5.1 The Main Algorithm

In this section, we describe and discuss the algorithm for solving the watchman route problem, thus (finally) arriving at the problem this article is aiming at to solve with reasonable run time.

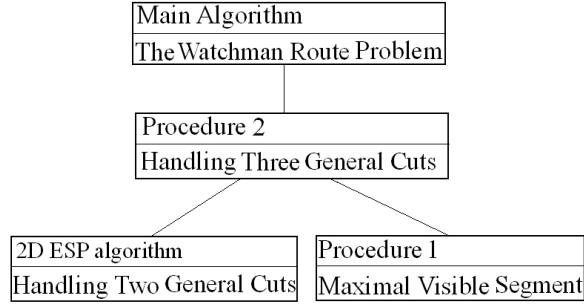
The used procedures are shown in Figure 10 where the main algorithm is based on Procedure 2, which applies the 2D ESP algorithm, and Procedure 1. We present the used procedures first and the main algorithm at the end.

As described in Section 4.1, the main idea of an RBA is as follows: In each iteration, we update (by finding a local minimum or optimal vertex) the second vertex p_i for every three-subsequent-vertices subsequence p_{i-1}, p_i, p_{i+1} in the step set $\{S_1, S_2, \dots, S_k\}$. The first procedure below computes the maximal visible segment which is actually a step element of the used RBA. The second procedure is used for updating the vertex.

Procedure 1 *Compute Maximal Visible Segment*

Input: Polygon Π and a general cut C of Π ; let v_1 and v_2 be two endpoints of C ; two points p and q such that $p \in C$ and p is a visible point of $q \in \partial\Pi \setminus C$.

Output: Two points $p'_1, p'_2 \in C$ such that $p \in \text{segment } p'_1p'_2$, and $p'_1p'_2$ is the maximal visible segment of q .

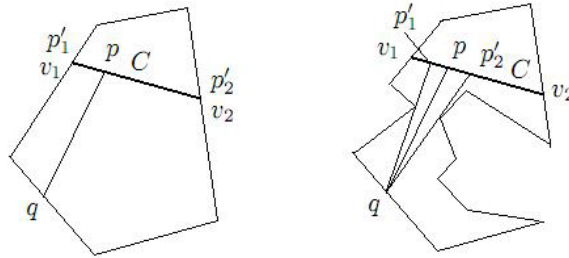
**Fig. 10.** The main algorithm and the used procedures.

At first we describe Procedure 1 informally as follows: In Case 1, we have that p is not an endpoint of C . For $i \in \{1, 2\}$, if q can see v_i , (see left, Figure 11), let $p'_i = v_i$; otherwise, let V_i be the set of vertices in $V(\partial\Pi)$ such that each vertex in V_i is inside of $\triangle qpv_i$. Apply the convex hull algorithm to compute $C(V_i)$. Apply the tangent algorithm to find a point $p'_i \in C$ such that qp'_i is a tangent to $C(V_i)$ (see right of Figure 11).

In Case 2, we have now that p is an endpoint of C . Without loss of generality, assume that $p = v_1$. Let $p'_1 = p$. Let V_2 be the set of vertices in $V(\partial\Pi)$ such that each vertex in V_2 is inside of $\triangle qpv_i$. Apply the convex hull algorithm to compute $C(V_2)$. Apply the tangent algorithm to find a point $p'_2 \in C$ such that qp'_2 is a tangent to $C(V_2)$.

The Pseudo code of Procedure 1 is also listed below:

- 1: **if** $p \notin \partial\Pi$ (i.e., p is not an endpoint of C) **then**
- 2: **for** $i \in \{1, 2\}$ **do**
- 3: **if** $qv_i \cap \partial\Pi = \emptyset$ (i.e., q can see v_i) (see left of Figure 11) **then**
- 4: Let $p'_i = v_i$.
- 5: **else**

**Fig. 11.** Illustration for Procedure 1.

- 6: Let V_i be the set of vertices in $V(\partial\Pi)$ such that each vertex in V_i is inside of $\triangle qpv_i$.
- 7: Apply the convex hull algorithm to compute $C(V_i)$.
- 8: Apply the tangent algorithm to find a point $p'_i \in C$ such that qp'_i is a tangent to $C(V_i)$ (see right of Figure 11).
- 9: **end if**
- 10: **end for**
- 11: **else**
- 12: Without loss of generality, assume that $p = v_1$. Let $p'_1 = p$.
- 13: Proceed analogously as in Steps 4–8, but now for $i = 2$.
- 14: **end if**
- 15: Output p'_i , $i = 1, 2$, and Stop.

Procedure 2 *Handling of Three General Cuts*

Input: Three general cuts C_1 , C_2 and C_3 of Π ; three points $p_i \in C_i$, where $i = 1, 2, 3$.

Output: An updated shorter path $\rho(p_1, \dots, p_2, \dots, p_3)$, which might also contain vertices of the polygon Π .

- 1: For both $i \in \{1, 2\}$, let $\{p_i, p_{i+1}\}$ (where $p_i \in C_i$) be the input for the 2D ESP algorithm; the output is a set V_{ii+1} . Let $V = V_{12} \cup V_{23}$.
- 2: Find q_1 and $q_3 \in V$ such that $\langle q_1, p_2, q_3 \rangle$ is a subsequence of V (i.e., q_1, p_2, q_3 appear consecutively in V ; see Figure 12).
- 3: Apply Procedure 1 to find the maximal visible segment s_i of q_i , $i = 1, 3$.
- 4: Find vertex $p'_2 \in s_2 = s_1 \cap s_3$ (see Figure 13) such that

$$d_e(q_1, p'_2) + d_e(p'_2, q_3) = \min\{d_e(q_1, p') + d_e(p', q_3) : p' \in s_2\}$$

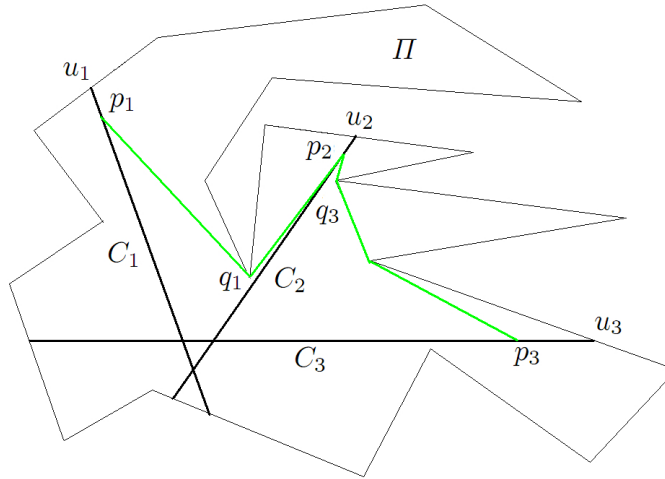


Fig. 12. Illustration for Step 2 of Procedure 2.

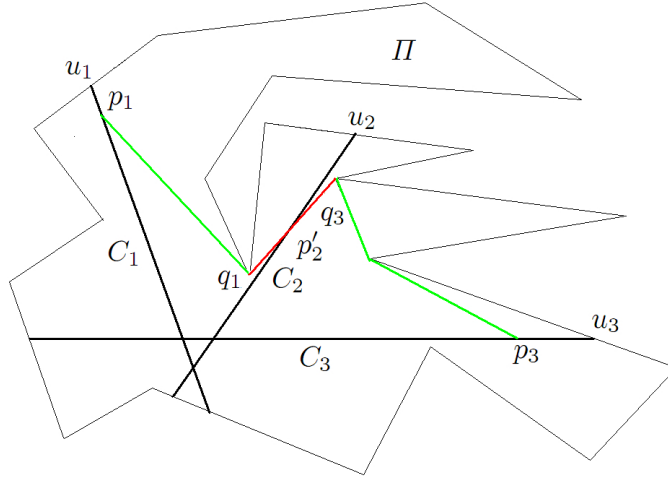


Fig. 13. Illustration for Step 4 of Procedure 2.

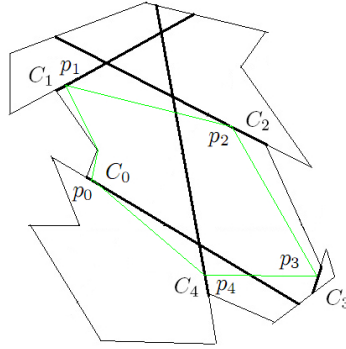


Fig. 14. Illustration for Step 1 of Algorithm 3.

- 5: If $C_2 \cap C_1$ (or C_3) $\neq \emptyset$ and p'_2 is the intersection point, then ε_2 -transform p'_2 into another point (still denoted by p'_2) in C_2 .
- 6: Update V by letting $p_2 = p'_2$.

We call the line segment s_2 in Step 4 of Procedure 2 *associated to* the updated (optimal) point p_2 .

Algorithm 3 *Main Algorithm for Solving the WRP*

Input: A degeneration accuracy constant $\varepsilon_2 > 0$; k essential cuts C_0, C_1, \dots, C_{k-1} , and Π , which satisfy the condition of the WRP, and points $p_i \in C_i$, where $i = 0, 1, 2, \dots, k-1$.

Output: An updated closed $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation path (i.e., the “route”) $\rho(p_0, \dots, p_1, \dots, p_{k-1})$, which may also contain vertices of polygon Π .

The following pseudo code is fairly easy to read, and we defer from providing another (more informal) high level description of Algorithm 3.

- 1: For each $i \in \{0, 1, \dots, k-1\}$, let p_i be a point in C_i (see Figure 14).
- 2: Let $V_0 = V = \langle p_0, p_1, \dots, p_{k-1} \rangle$; $L_1 = \sum_{i=0}^{k-1} L_{\Pi}(p_i, p_{i+1})$; and $L_0 = 0$.
- 3: **while** $L_1 - L_0 > \varepsilon$ **do**
- 4: **for** each $i \in \{0, 1, \dots, k-1\}$ **do**
- 5: Let C_{i-1} , C_i , C_{i+1} , p_{i-1} , p_i , p_{i+1} and Π be the input for Procedure 2, which updates p_i in V_0 .
- 6: Let U_i be the sequence of vertices of the path $\rho(p_{i-1}, \dots, p_i, \dots, p_{i+1})$ with respect to C_{i-1} , C_i and C_{i+1} (inside of Π); let $U_i = \langle q_1, q_2, \dots, q_m \rangle$.
- 7: Insert (after p_{i-1}) the points of sequence U_i (in the given order) into V_0 ; that means, we have that

$$V_1 = \langle p_0, p_1, \dots, p_{i-1}, q_1, q_2, \dots, q_m, p_{i+1}, \dots, p_{k-1} \rangle$$

(note: sequence V_1 is the updated sequence V_0 , after inserting U_i)

- 8: **end for**
- 9: Let $L_0 = L_1$ and $V_0 = V$ (note: we use the updated original sequence V instead of V_1 for the next iteration).
- 10: Calculate the perimeter L_1 of the polygon, given by the sequence V_1 of vertices.
- 11: **end while**
- 12: Output sequence V_1 , and the desired length equals to L_1 .

5.2 Correctness and Time Complexity

The following basic results of convex analysis may be found, for example, in [6, 44, 45]:

Theorem 4. ([45], Theorem 3.5) *Let S_1 and S_2 be convex sets in \mathbb{R}^m and \mathbb{R}^n , respectively. Then $S_1 \times S_2$ is a convex set in \mathbb{R}^{m+n} , where $m, n \in \mathbb{N}$.*

Proposition 2. ([6], page 27) *Each line segment is a convex set.*

Proposition 3. ([6], page 72) *Each norm on \mathbb{R}^n is a convex function.*

Proposition 4. ([6], page 79) *A nonnegative weighted sum of convex functions is a convex function.*

Proposition 5. ([45], page 264) *Let f be a convex function. If x is a point where f has a finite local minimum, then x is a point where f has its global minimum.*

By Theorem 4 and Propositions 2–4, we have the following

Observation 6 $L_S(p, q): s_0 \times s_{k-1} \rightarrow \mathbb{R}$ *is a convex function.*

Let C_1 , C_2 , and Π satisfy the condition of the watchman route problem. By Observation 6, we have the following

Observation 7 $L_\Pi(p, q): C_1 \times C_2 \rightarrow \mathbb{R}$ is a convex function.

Let $s_i \subseteq C_i$ be the line segment associated to the final updated point $p_i \in C_i$ in Algorithm 3, where $i = 0, 1, 2, \dots, k-1$. Analogous to Theorem 3, we have the following

Theorem 5. *If Assumption ?? is valid and the chosen accuracy constant ε is sufficiently small, then Algorithm 3 outputs a unique $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation (closed) path with respect to the step set $\langle s_0, s_1, \dots, s_{k-1}, s_0 \rangle$, for any initial path.*

Regarding the proof of correctness for Theorem 5, at first we define that Algorithm 1 is also called an *arc version* of an RBA. If we modify Algorithm 1 such that p and q are not specified by finding a shortest closed path which passes through line segments $\langle S_1, S_2, \dots, S_k, S_1 \rangle$ in order, then we obtain the *curve version* of Algorithm 1.

Basically, following the same way as demonstrated with the proof of Theorem 3, we can prove that the curve version of Algorithm 1 outputs a closed $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation path. Thus, Algorithm 3 defines a closed $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation path to the step set $\langle s_0, s_1, \dots, s_{k-1}, s_0 \rangle$, but we will not provide the proof here due to given similarities and available space. Theorem 5 indicates that Algorithm 3 outputs an approximate local minimal solution to the WRP, then we have the following

Theorem 6. *Algorithm 3 outputs an $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation solution to the WRP.*

Proof. By Observation 7,

$$\sum_{i=0}^{k-1} L_\Pi(p_i, p_{i+1}) : \prod_{i=0}^{k-1} C_i \rightarrow \mathbb{R}$$

is a convex function, where

$$L_\Pi(p_i, p_{i+1})$$

is defined as in Step 2 of Algorithm 3. By Proposition 5 and Theorem 5, we have proved the theorem. \square

Regarding the time complexity of our solution to the WRP, we first state two observations (where proofs are mathematically very trivial and are skipped here for this reason);

Observation 8 *Procedure 1 can be computed in time $\mathcal{O}(|V(\partial\Pi)|)$.*

Observation 9 *Procedure 2 can be computed in time $\mathcal{O}(|V(\partial\Pi)|)$.*

Furthermore, note that the main computation is in the two stacked loops. The while-loop takes $\kappa(\varepsilon)$ iterations. By Observation 9, the for-loop can be computed in time $\mathcal{O}(k \cdot |V(\partial\Pi)|)$. Thus,

Observation 10 *Algorithm 3 can be computed in time $\kappa(\varepsilon) \cdot \mathcal{O}(k \cdot |V(\partial\Pi)|)$.*

By Lemma 1, Theorem 2, and Observation 10, we have the following

Theorem 7. *This paper provided an $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation solution to the WRP, having time complexity $\kappa(\varepsilon) \cdot \mathcal{O}(k|V(\partial\Pi)|)$, where k is the number of essential cuts, and L is the length of an optimal watchman route.*

6 Solving the Safari Route Problem

6.1 The Algorithm

In this subsection, we describe an algorithm for solving the SRP. It is modified from Algorithm 3. All modifications are underlined. We start presenting a procedure frequently used by the algorithm.

Procedure 3 *Handling of Three Polygons*

Input: A degeneration accuracy constant $\varepsilon_2 > 0$ and a constant $\varepsilon_3 \geq \varepsilon_2$; four polygons P_1, P_2, P_3 and Π satisfy the SRP condition; ε_3 -polygon of P_2 , denoted by P'_2 ; three points $p_i \in \partial P_i$, where $i = 1, 2, 3$.

Output: An updated shorter path $\rho(p_1, \dots, p_2, \dots, p_3)$, which might also contain vertices of the polygon Π .

- 1: For both $i \in \{1, 2\}$, let $\{p_i, p_{i+1}\}$ (where $p_i \in \partial P_i$) be the input for the 2D ESP algorithm; the output is a sequence of vertices V_{ii+1} of $\rho_\Pi(p_i, p_{i+1})$. Let V be $V_{12} \cup V_{23}$.
- 2: Find $q_1 = \rho_\Pi(p_1, p_2) \cap \partial P'_2$; and $q_3 = \rho_\Pi(p_2, p_3) \cap \partial P'_2$ (see Figure 15).
- 3: Let e be the edge of P_2 such that e is contained on the boundary of Π ; e' e 's corresponding edge in P'_2 . Let u and v (u' and v') be the two endpoints of e (e') (see Figure 15). Let u'' (v'') be a point in the line segment uu' (vv') such that $d_e(u'', u) = \varepsilon_2$ ($d_e(v'', v) = \varepsilon_2$) (see Figure 15). If q_1 or q_3 is located on segment uu'' or vv'' , then reset it to be u'' or v'' .
- 4: If $q_1 q_3 \cap P'_2 \neq \emptyset$, then let p'_2 be the point in segment $q_1 q_3 \cap P'_2$ such that

$$d_e(q_1, p'_2) = \min\{d_e(q_1, p') : p' \in q_1 q_3 \cap P'_2\}$$

Otherwise, find vertex $p'_2 \in \partial P'_2$ (see Figure 15) such that

$$d_e(q_1, p'_2) + d_e(p'_2, q_3) = \min\{d_e(q_1, p') + d_e(p', q_3) : p' \in \partial P'_2\}$$

- 5: Update V by letting $p_2 = p'_2$.

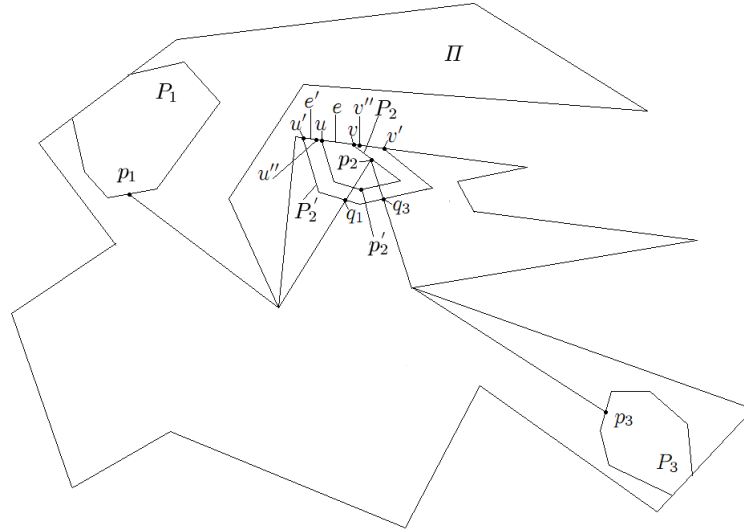


Fig. 15. Illustration for Procedure 3.

Algorithm 4 *Main Algorithm for Solving the SRP*

Input: A degeneration accuracy constant $\varepsilon_2 > 0$ and a constant $\varepsilon_3 \geq \varepsilon_2$; k convex polygons P_0, P_1, \dots, P_{k-1} , and Π , which satisfy the condition of the SRP, and points $p_i \in \partial P_i$, where $i = 0, 1, 2, \dots, k-1$.

Output: An updated closed $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation path (i.e., the “route”) $\rho(p_0, \dots, p_1, \dots, p_{k-1})$, which may also contain vertices of polygon Π .

- 1: For each $i \in \{0, 1, \dots, k-1\}$, let p_i be a point in ∂P_i .
- 2: Let $V_0 = V = \langle p_0, p_1, \dots, p_{k-1} \rangle$; $L_1 = \sum_{i=0}^{k-1} L_\Pi(p_i, p_{i+1})$; and $L_0 = 0$.
- 3: **while** $L_1 - L_0 > \varepsilon$ **do**
- 4: **for** each $i \in \{0, 1, \dots, k-1\}$ **do**
- 5: Let $P_{i-1}, P_i, P_{i+1}, p_{i-1}, p_i, p_{i+1}$ and Π be the input for Procedure 3, which updates p_i in V_0 .
- 6: Let U_i be the sequence of vertices of the path $\rho(p_{i-1}, \dots, p_i, \dots, p_{i+1})$ with respect to P_{i-1}, P_i and P_{i+1} (inside of Π); let $U_i = \langle q_1, q_2, \dots, q_m \rangle$.
- 7: Insert (after p_{i-1}) the points of sequence U_i (in the given order) into V_0 ; that means, we have that

$$V_1 = \langle p_0, p_1, \dots, p_{i-1}, q_1, q_2, \dots, q_m, p_{i+1}, \dots, p_{k-1} \rangle$$

(note: sequence V_1 is the updated sequence V_0 , after inserting U_i)

- 8: **end for**
- 9: Let $L_0 = L_1$ and $V_0 = V$ (note: we use the updated original sequence V instead of V_1 for the next iteration).

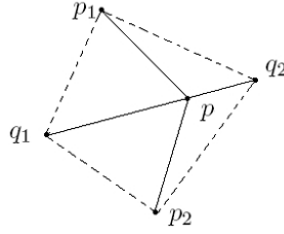


Fig. 16. Illustration for the proof of Observation 11.

- 10: Calculate the perimeter L_1 of the polygon, given by the sequence V_1 of vertices.
- 11: **end while**
- 12: Output sequence V_1 , and the desired length equals to L_1 .

6.2 Correctness and Time Complexity

Let q_1q_2 be a line segment. Let p_1 and p_2 be two points outside of q_1q_2 and $d_e(p_1, q_1) + d_e(p_2, q_1) = d_e(p_1, q_2) + d_e(p_2, q_2)$. Let p be a point on the straight line which contains segment q_1q_2 , denoted by $l_{p_1p_2}$, such that

$$d_e(p_1, p) + d_e(p_2, p) = \min\{d_e(p_1, q) + d_e(p_2, q) : q \in l_{q_1q_2}\}$$

(see Figure 16)

Then we have the following

Observation 11 p is in between q_1 and q_2 .

Proof. Without loss of generality, suppose that q_1q_2 is parallel to the x -axis. Let the coordinates of p_i be (a_i, b_i, c_i) , where $i = 1, 2$. Let $p = (x, b, c)$ be a point on the line $l_{q_1q_2}$. Then,

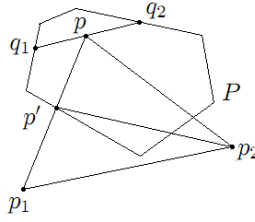
$$d_e(p_i, p) = \sqrt{(x - a_i)^2 + (b - b_i)^2 + (c - c_i)^2}$$

for $i = 1, 2$. Let $f(x) = d_e(p_1, p) + d_e(p_2, p)$. Then we have that

$$f'(x) = \frac{x - a_1}{\sqrt{(x - a_1)^2 + (b - b_1)^2 + (c - c_1)^2}} + \frac{x - a_2}{\sqrt{(x - a_2)^2 + (b - b_2)^2 + (c - c_2)^2}}$$

By setting $f'(x) = 0$, we can find a unique critical point x_p of function $f(x)$; that means, the coordinates of p are equal to (x_p, b, c) . Let the coordinates of q_i be equal to (a_{q_i}, b, c) , where $i = 1, 2$. Since x_p is the unique critical point of the function $f(x)$, it follows that $f(x)$ is decreasing in the interval $(-\infty, x_p)$ and increasing in the interval (x_p, ∞) . Because

$$d_e(p_1, q_1) + d_e(p_2, q_1) = d_e(p_1, q_2) + d_e(p_2, q_2)$$

**Fig. 17.** Illustration for the proof of Observation 12.

implies $f(a_{q_1}) = f(a_{q_2})$, we have that $a_{q_1} \in (-\infty, x_p)$ and $a_{q_2} \in (x_p, \infty)$. Thus, x_p is located between a_{q_1} and a_{q_2} . This proves the observation. \square

Let P be a convex region; ∂P the boundary of P ; p_1 and p_2 two points outside of P such that $P \cap p_1p_2 = \emptyset$, where p_1p_2 is the straight line passing through points p_1 and p_2 . Then we have the following

Observation 12 *There is a unique point q in ∂P such that*

$$d_e(p_1, q) + d_e(p_2, q) = \min\{d_e(p_1, q) + d_e(p_2, q) : q \in P\}$$

Proof. If p is a point in P such that

$$d_e(p_1, p) + d_e(p_2, p) = \min\{d_e(p_1, q) + d_e(p_2, q) : q \in P\} \quad (8)$$

Then p must be located on ∂P . Otherwise, there exists a point p' in $\triangle pp_1p_2 \cap \partial P$ such that

$$d_e(p_1, p') + d_e(p_2, p') < d_e(p_1, p) + d_e(p_2, p)$$

This is a contradiction to Equation (8) (see Figure 17).

Suppose that there exist two points q_i in ∂P such that

$$d_e(p_1, q_i) + d_e(p_2, q_i) = \min\{d_e(p_1, q) + d_e(p_2, q) : q \in P\}$$

where $i = 1, 2$. By Observation 11, there exists a point p in between q_1 and q_2 such that

$$d_e(p_1, p) + d_e(p_2, p) < \min\{d_e(p_1, q) + d_e(p_2, q) : q \in P\} \quad (9)$$

As P is a convex region, and q_1, q_2 are in ∂P , p must be in $P \setminus \partial P$. Then there exists a point p' in $\triangle pp_1p_2 \cap \partial P$ such that

$$d_e(p_1, p') + d_e(p_2, p') < d_e(p_1, p) + d_e(p_2, p)$$

By Inequality (9), we have that

$$d_e(p_1, p') + d_e(p_2, p') < \min\{d_e(p_1, q) + d_e(p_2, q) : q \in P\}$$

This is a contradiction. \square

Theorem 8. *Algorithm 4 outputs an $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation solution to the SRP.*

Proof. Algorithm 4 defines a function f mapping from $\prod_{i=0}^{k-1} \partial P_i$ to \mathbb{R} . Analogous to the proof of Observation 5, the total values of f are finite. By Observation 12, f is a continuous function. Thus, f must be a singleton. By Lemma 2, this proves the theorem. \square

Regarding the time complexity of our solution to the SRP, note that the main computation is in the two stacked loops. The while-loop takes $\kappa(\varepsilon)$ iterations. Procedure 3 can be computed in time $\mathcal{O}(|V(\partial\Pi)| + |V(\partial P_i)|)$. So the for-loop can be computed in time $\mathcal{O}(k \cdot |V(\partial\Pi)| + \sum_{i=0}^{k-1} |V(\partial P_i)|)$. Thus,

Observation 13 *Algorithm 4 can be computed in time $\kappa(\varepsilon) \cdot \mathcal{O}(k \cdot |V(\partial\Pi)| + \sum_{i=0}^{k-1} |V(\partial P_i)|)$.*

By Lemma 2, Theorem 8, and Observation 13, we have the following

Theorem 9. *This paper provided an $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation solution to the SRP, having time complexity $\kappa(\varepsilon) \cdot \mathcal{O}(k|V(\partial\Pi)| + \sum_{i=0}^{k-1} |V(\partial P_i)|)$, where k is the number of convex polygons, and L is the length of an optimal safari route.*

7 Solving the Zookeeper Route Problem

In this section, we modify the algorithms and theorems in Section 6 for solving the ZRP. All modifications are underlined>.

7.1 The Algorithm

Procedure 4 *Handling of Three Polygons*

Input: A degeneration accuracy constant $\varepsilon_2 > 0$ and a constant $\varepsilon_3 \geq \varepsilon_2$; four polygons P_1, P_2, P_3 and Π satisfy the ZRP condition; ε_3 -polygon of P_2 , denoted by P'_2 ; three ZRP pair $(p_{i_1}, p_{i_2}) \in \partial P_i$, where $i = 1, 2, 3$.

Output: An updated shorter path $\rho(p_{1_1}, \dots, p_{1_2}, \dots, p_{2_1}, \dots, p_{2_2}, \dots, p_{3_1}, \dots, p_{3_2})$, which might also contain vertices of the polygon Π .

- 1: For both $i \in \{1, 2\}$, let $\{p_{i_2}, p_{i+1_1}\}$ (where $p_{i_2} \in \partial P_i, p_{i+1_1} \in \partial P_{i+1}$) be the input for the 2D ESP algorithm; the output is a sequence of vertices V_{ii+1} of $\rho_{\Pi'}(p_{i_2}, p_{i+1_1})$. Let V be $V_{12} \cup V_{23}$.
- 2: Find $q_1 = \rho_{\Pi'}(p_{1_2}, p_{2_1}) \cap \partial P'_2$; and $q_3 = \rho_{\Pi'}(p_{2_2}, p_{3_1}) \cap \partial P'_2$ (see Figure 18).
- 3: Let e be the edge of P_2 such that e is contained on the boundary of Π ; e' e 's corresponding edge in P'_2 . Let u and v (u' and v') be the two endpoints of e (e') (see Figure 18). Let u'' (v'') be a point in the line segment uu' (vv') such that $d_e(u'', u) = \varepsilon_2$ ($d_e(v'', v) = \varepsilon_2$) (see Figure 18). If q_1 or q_3 is located on segment uu'' or vv'' , then reset it to be u'' or v'' .

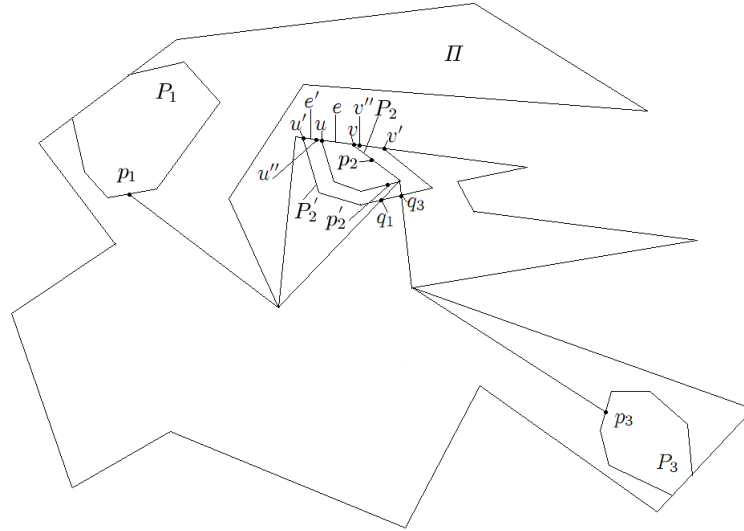


Fig. 18. Illustration for Procedure 4, where $p_i = p_{i_1} = p_{i_2}$ ($i = 1, 2, 3$), and $p'_2 = p'_{2_1} = p'_{2_2}$.

- 4: If $q_1 q_3 \cap P'_2 \neq \emptyset$, then apply the tangent algorithm to find two points $p'_{2_1}, p'_{2_2} \in \partial P_2$ such that $q_1 p'_{2_1} (q_3 p'_{2_2})$ is a tangent to P_2 , and p'_{2_1}, p'_{2_2} are ordered anti-clockwise around ∂P_2 .
 Otherwise, find vertex $p'_2 \in \partial P'_2$ (see Figure 18) such that

$$d_e(q_1, p'_2) + d_e(p'_2, q_3) = \min\{d_e(q_1, p') + d_e(p', q_3) : p'_2 \in \partial P'_2\}$$

let p'_{2_1} and p'_{2_2} be p'_2 .

- 5: Update V by letting $(p_{2_1}, p_{2_2}) = (p'_{2_1}, p'_{2_2})$.

Algorithm 5 *Main Algorithm for Solving the ZRP*

Input: A degeneration accuracy constant $\varepsilon_2 > 0$ and a constant $\varepsilon_3 \geq \varepsilon_2$; k convex polygons P_0, P_1, \dots, P_{k-1} , and Π , which satisfy the condition of the ZRP, and ZRP pair $(p_{i_1}, p_{i_2}) \in \partial P_i$, where $i = 0, 1, 2, \dots, k-1$.

Output: An updated closed $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation path (i.e., the “route”) $\rho(p_{0_1}, \dots, p_{0_2}, \dots, p_{1_1}, \dots, p_{1_2}, \dots, p_{k-1_1}, \dots, p_{k-1_2})$, which may also contain vertices of polygon Π .

- 1: For each $i \in \{0, 1, \dots, k-1\}$, let (p_{i_1}, p_{i_2}) be a ZRP pair in ∂P_i .

- 2: Let $V_0 = V = \langle p_{0_1}, \dots, p_{0_2}, \dots, p_{1_1}, \dots, p_{1_2}, \dots, p_{k-1_1}, \dots, p_{k-1_2} \rangle$;

$$L_1 = \sum_{i=0}^{k-1} L_{\Pi}(p_{i_1}, p_{i+1_2}); \text{ and } L_0 = 0.$$

- 3: **while** $L_1 - L_0 > \varepsilon$ **do**
- 4: **for** each $i \in \{0, 1, \dots, k-1\}$ **do**
- 5: Let $P_{i-1}, P_i, P_{i+1}, \underline{(p_{i-1_1}, p_{i-1_2})}, (p_{i_1}, p_{i_2}), (p_{i+1_1}, p_{i+1_2})$ and Π be the input for Procedure 4, which updates $\underline{(p_{i_1}, p_{i_2})}$ in V_0 .
- 6: Let U_i be the sequence of vertices of the path $\underline{\rho(p_{i-1_1}, \dots, p_{i-1_2}, \dots, p_{i_1}, \dots, p_{i_2}, \dots, p_{i+1_1}, \dots, p_{i+1_2})}$ with respect to P_{i-1}, P_i and P_{i+1} (inside of Π); let $U_i = \langle q_1, q_2, \dots, q_m \rangle$.
- 7: Insert (after $\underline{p_{i-1_2}}$) the points of sequence U_i (in the given order) into V_0 ; that means, we have that

$$\underline{V_1 = \langle p_{0_1}, \dots, p_{0_2}, \dots, p_{1_1}, \dots, p_{1_2}, \dots, p_{i-1_1}, \dots, p_{i-1_2},$$

$$q_1, q_2, \dots, q_m,$$

$$\underline{p_{i+1_1}, \dots, p_{i+1_2}, \dots, p_{k-1_1}, \dots, p_{k-1_2} \rangle}$$

(note: sequence V_1 is the updated sequence V_0 , after inserting U_i)

- 8: **end for**
- 9: Let $L_0 = L_1$ and $V_0 = V$ (note: we use the updated original sequence V instead of V_1 for the next iteration).
- 10: Calculate the perimeter L_1 of the polygon, given by the sequence V_1 of vertices.
- 11: **end while**
- 12: Output sequence V_1 , and the desired length equals to L_1 .

7.2 Correctness and Time Complexity

Analogous to the proof of Theorem 8, we can prove that

Theorem 10. *Algorithm 5 outputs an $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation solution to the ZRP.*

And analogous to the proof of Theorem 9, we can prove that

Theorem 11. *This paper provided an $\{1 + 4k \cdot [r(\varepsilon) + \varepsilon_2]/L\}$ -approximation solution to the ZRP, having time complexity $\kappa(\varepsilon) \cdot \mathcal{O}(k(|V(\partial\Pi)| + \sum_{i=0}^{k-1} |V(\partial P_i)|))$, where k is the number of convex polygons, and L is the length of an optimal zookeeper route.*

8 Some Generalization

So far, almost all algorithms for solving various Euclidean shortest path problems (ESP) only consider the environment containing geometric objects which have straight edges. For example, the obstacles of ESP are modeled by polygons in 2D; polyhedrons in 3D; and the surface of polyhedrons in 2.5D [37]. The obstacles with smooth surfaces started becoming interesting in computational geometry [38]. In this section, we discuss some generalization of the SRP and ZRP, where each convex polygon is replaced by a simple region.

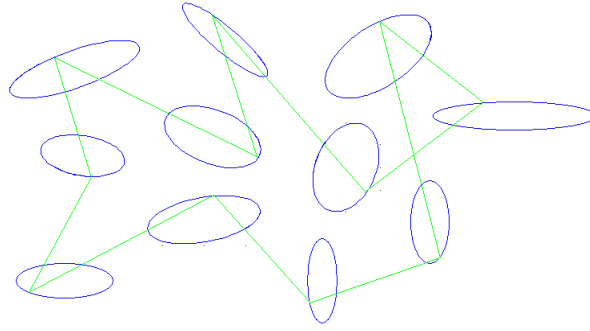


Fig. 19. An example of touring a sequence of ellipses.

8.1 Touring Ellipses Problem

In the SRP, if the simple polygon Π is removed and all convex polygons P_i s are replaced by simple polygons, then such refined SRP is known as touring polygons problem (TPP) which was discussed in [22]. In this subsection, we refine the SRP such that the simple polygon Π is removed and all convex polygons P_i s are replaced by ellipses (see Figure 19). We call such problem *touring ellipses problem (TEP)*. Let E_i s be pairwise disjoint ellipses, where $i = 0, 1, \dots, k-1$; E_i° the interior of E_i ; ∂E_i the boundary of E_i .

Calculate a shortest route ρ such that ρ visits each ellipse E_i at a point p_i on ∂E_i (the boundary of E_i), allowing that straight segments $p_i p_{i+1} \pmod k$ intersect E_i° in one or several segments, for $i = 0, 1, \dots, k-1$.

We describe the following algorithm for solving the TEP. It is modified from Algorithm 1.

Algorithm 6 for Solving the TEP

Input: A sequence of k pairwise disjoint ellipses E_0, E_1, \dots, E_{k-1} and an accuracy constant $\varepsilon > 0$.

Output: A sequence $\langle p_0, p_1, p_2, \dots, p_{k-1}, p_0 \rangle$ of an $[1+4(k+1)r(\varepsilon)/L]$ -approximation path which starts at p_0 , then visits (i.e., passes through) ∂E_i at p_i in the given order, and finally ends at p_0 .

- 1: For each $i \in \{0, 1, \dots, k-1\}$, let p_i be a point in ∂E_i .
- 2: Calculate $L_1 = \sum_{i=0}^{k-1} d_e(p_i, p_{i+1})$; and let L_0 be 0.
- 3: **while** $L_1 - L_0 > \varepsilon$ **do**
- 4: **for** each $i \in \{0, 1, \dots, k-1\}$ **do**
- 5: Compute a point $q_i \in \partial E_i$ such that

$$d_e(p_{i-1}, q_i) + d_e(q_i, p_{i+1}) = \min\{d_e(p_{i-1}, p) + d_e(p, p_{i+1}) : p \in \partial E_i\}$$
- 6: Update the path $\langle p_0, p_1, p_2, \dots, p_{k-1} \rangle$ by replacing p_i by q_i .
- 7: **end for**
- 8: Let $L_0 = L_1$ and calculate $L_1 = \sum_{i=0}^{k-1} d_e(p_i, p_{i+1})$.

9: **end while**

10: Return $\langle p_0, p_1, p_2, \dots, p_{k-1}, p_0 \rangle$.

The optimal point q_i in Step 5 can be computed as follows: Each point p on ∂E_i can be expressed as $(x(t), y(t))$, where $x(t) = x_c + a \cos t \cos \theta - b \sin t \sin \theta$ and $y(t) = y_c + a \cos t \sin \theta + b \sin t \cos \theta$, the parameter t varies from 0 to 2π , (x_c, y_c) is the center of the ellipse E_i , a and b are one-half of E_i 's major and minor axes respectively, and θ is the angle between the X-axis and the major axis of E_i . Therefore, the coordinates of p have the form

$$(a_1 \cos t + b_1 \sin t + c_1, a_2 \cos t + b_2 \sin t + c_2)$$

where $t \in [0, 2\pi]$, a_i , b_i and c_i ($i = 1, 2$) are functions of x_c , y_c , a , b , and θ . Let $p_{i-1} = (x_1, y_1)$, $p_{i+1} = (x_2, y_2)$. The optimal point q_i is decided by an optimal value of t which is a solution of the following equation

$$\frac{\partial d_e(p_{i-1}, p) + d_e(p, p_{i+1})}{\partial t} = 0$$

Replace p , p_{i-1} and p_{i+1} by their coordinates in this equation, and by The Distance Formula, we obtain the following equation for deciding the optimal t :

$$\sum_{j=1}^6 (A_j \sin jt + B_j \cos jt) + A_0 = 0 \quad (10)$$

where A_j , B_j ($j = 1, 2, \dots, 6$) and A_0 are functions of x_c , y_c , a , b , x_1 , y_1 , x_2 , and y_2 . Equation (10) can be rewritten as:

$$\sum_{j=0}^{12} C_j x^j = 0 \quad (11)$$

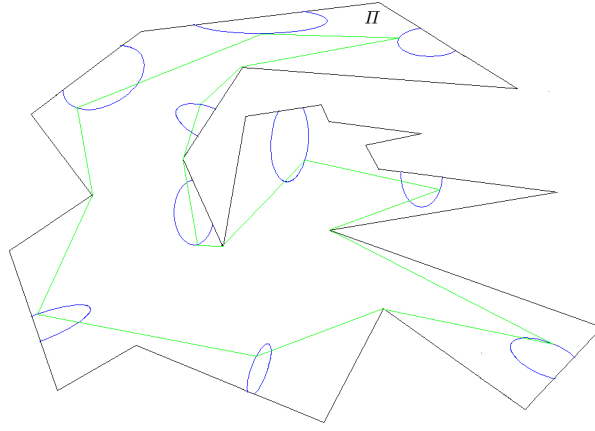
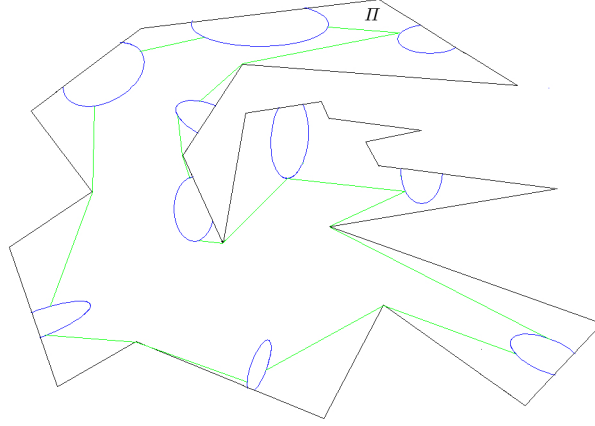
where $x = \tan \frac{t}{2}$, C_j ($j = 1, 2, \dots, 12$) are functions of x_c , y_c , a , b , x_1 , y_1 , x_2 , and y_2 .

Equation (11) can be solved in time $\mathcal{O}(1)$ using a QR Algorithm [58]. Thus, analogous to the analysis in Section 4.4, and the proof of Theorem 8, we have that

Theorem 12. *Algorithm 6 outputs an $\{1 + 4k \cdot \lceil r(\varepsilon) \rceil / L\}$ -approximation solution to the TEP in time $\kappa(\varepsilon) \mathcal{O}(k)$.*

8.2 The General SRP and ZRP

In the SRP (ZRP), if each convex is replaced by a simple region, then we say such problem is the *general SRP* (ZRP). For example, Figure (20) (Figure (21)) shows an initial route of the general SRP (ZRP). It is clear that Algorithm 4 (Algorithm 5) can be modified to solve such SRP (ZRP) with the same time complexity.

**Fig. 20.** An example of the general SRP.**Fig. 21.** An example of the general ZRP.

9 Concluding Remarks

In this paper, we have recalled the basic idea of the first RBA, which was proposed in digital geometry [9, 30]. We refined this useful idea such that it becomes a simple “arc” version of an RBA; see Algorithm 1. In particular, a contribution in this paper is the provided simple and efficient way for solving a system formed by partial differential Equations (5). Algorithm 1 runs in time $\kappa(\varepsilon) \cdot \mathcal{O}(k)$, while the solution proposed by [46] has a time complexity which is doubly exponential in k . Another contribution in this paper is that the basic idea of a RBA can be generalized to establish a whole class of rubberband algorithms (RBAs) for solving various Euclidean shortest path problems. The main algorithm of this paper (Algorithm 3) is just an example for such RBAs. It is not only significantly faster but also significantly simpler, easier to understand and to implement than

already published algorithms for solving the WRP, SRP and ZRP. Finally, we are the first to provide algorithms for solving more general SRP and ZRP where all convex polygons are replaced by simple regions such as convex polybeziers or ellipses.

References

1. M. H. Alsuwaiyel and D. T. Lee. Minimal link visibility paths inside a simple polygon. *Comput. Geom.*, **3**(1): 1–25, 1993.
2. M. H. Alsuwaiyel and D. T. Lee. Finding an approximate minimum-link visibility path inside a simple polygon. *Information Processing Letters*, **55**:75–79, 1995.
3. E. M. Arkin, J. S. B. Mitchell, and C. Piatko. Minimum-link watchman tours. Report, University at Stony Brook, 1994.
4. T. Asano, S. K. Ghosh, and T. C. Shermer. Visibility in the plane. In *Handbook of Computational Geometry* (J.-R. Sack and J. Urrutia, editors), pages 829–876, Elsevier, 2000.
5. S. Bespamyatnikh. An $\mathcal{O}(n \log n)$ algorithm for the zoo-keepers problem. *Computational Geometry: Theory and Applications*, **24**:63–74, 2003.
6. S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, UK, 2004.
7. T. Bülow and R. Klette. Rubber band algorithm for estimating the length of digitized space-curves. In *Proc. Intern. Conf. Pattern Recognition*, volume 3, pages 551–555, 2000.
8. T. Bülow and R. Klette. Approximation of 3D shortest polygons in simple cube curves. In *Proc. Digital and Image Geometry*, LNCS 2243, pages 281–294, Springer, Berlin, 2001.
9. T. Bülow and R. Klette. Digital curves in 3D space and a linear-time length estimation algorithm. *IEEE Trans. Pattern Analysis Machine Intelligence*, **24**:962–970, 2002.
10. S. Carlsson, H. Jonsson, and B. J. Nilsson. Optimum guard covers and m -watchmen routes for restricted polygons. *Proc. Workshop Algorithms Data Struct.*, LNCS 519, pages 367–378, Springer, 1991.
11. S. Carlsson, H. Jonsson, and B. J. Nilsson. Optimum guard covers and m -watchmen routes for restricted polygons. *Int. J. Comput. Geom. Appl.*, **3**:85–105, 1993.
12. S. Carlsson and H. Jonsson. Computing a shortest watchman path in a simple polygon in polynomial-time. *Proc. Workshop Algorithms Data Struct.*, LNCS 955, pages 122–134, Springer, 1995.
13. S. Carlsson, H. Jonsson, and B. J. Nilsson. Approximating the shortest watchman route in a simple polygon. Technical report, Lund University, Sweden, 1997.
14. S. Carlsson, H. Jonsson, and B. J. Nilsson. Finding the shortest watchman route in a simple polygon. *Discrete Computational Geom.*, **22**:377–402, 1999.
15. B. Chazelle. Triangulating a simple polygon in linear time. *Discrete Computational Geometry*, **6**:485–524, 1991.
16. W.-P. Chin and S. Ntafos. Optimum zookeeper routes. *Combinatorics, Graph Theory and Computing*. In *Proc. South-East Conf., Boca Raton.*, Congr. Number **58**, 257–266, 1987.
17. W. Chin and S. Ntafos. Optimum watchman routes. *Information Processing Letters*, **28**:39–44, 1988.

18. W.-P. Chin and S. Ntafos. Shortest watchman routes in simple polygons. *Discrete Computational Geometry*, **6**:9–31, 1991.
19. W.-P. Chin and S. Ntafos. The zookeeper route problem. *Information Sciences*, **63**:245–259, 1992.
20. J. Choi, J. Sellen, and C.-K. Yap. Precision-sensitive Euclidean shortest path in 3-space. In Proc. *Ann. ACM Symp. Computational Geometry*, pages 350–359, 1995.
21. J. Czyzowicz, P. Egyed, H. Everett, W. Lenhart, K. Lyons, D. Rappaport, T. Shermer, D. Souvaine, G. Toussaint, J. Urrutia, and S. Whitesides. The aquarium keeper’s problem. In Proc. *ACM-SIAM Sympos. Data Structures Algorithms*, pages 459–464, 1991.
22. M. Dror, A. Efrat, A. Lubiw, and J. Mitchell. Touring a sequence of polygons. In Proc. *STOC*, pages 473–482, 2003.
23. L. P. Gewali, A. Meng., J. S. B. Mitchell, and S. Ntafos. Path planning in 0/1/infinity weighted regions with applications. *ORSA J. Comput.*, **2**:253–272, 1990.
24. L. P. Gewali and R. Lombardo. Watchman routes for a pair of convex polygons. *Lecture Notes in Pure Appl. Math.*, volume 144. 1993.
25. L. P. Gewali and S. Ntafos. Watchman routes in the presence of a pair of convex polygons. In Proc. *Canad. Conf. Comput. Geom.*, pages 127–132, 1995.
26. M. Hammar and B. J. Nilsson. Concerning the time bounds of existing shortest watchman routes. In Proc. *FCT’97*, LNCS 1279, pages 210–221, 1997.
27. J. Hersherberger and J. Snoeyink. An efficient solution to the zookeeper’s problem. In Proc. *Canad. Conf. Comput. Geom.*, pages 104–109, 1994.
28. D. S. Hochbaum (editor). *Approximation Algorithms for NP-Hard Problems*. PWS Pub. Co., Boston, 1997.
29. R. Klette and T. Bülow. Critical edges in simple cube-curves. In Proc. *Discrete Geometry Computational Imaging*, LNCS 1953, pages 467–478, Springer, Berlin, 2000.
30. R. Klette and A. Rosenfeld. *Digital Geometry*. Morgan Kaufmann, San Francisco, 2004.
31. P. Kumar and C. Veni Madhavan. Shortest watchman tours in weak visibility polygons. In Proc. *Canad. Conf. Comput. Geom.*, pages 91–96, 1995.
32. F. Li and R. Klette. Rubberband algorithms for solving various 2D or 3D shortest path problems. Invited talk, in IEEE Proc. *Computing: Theory and Applications*, The Indian Statistical Institute, Kolkata, pages 9 - 18, 2007.
33. H. Luo and A. Eleftheriadis. Rubberband: an improved graph search algorithm for interactive object segmentation. In Proc. *Int. Conf. Image Processing*, volume 1, pages 101–104, 2002.
34. C. Mata and J. S. B. Mitchell. Approximation algorithms for geometric tour and network design problems. In Proc. *Ann. ACM Sympos. Comput. Geom.*, pages 360–369, 1995.
35. A. Melkman. On-line construction of the convex hull of a simple polygon. *Information Processing Letters*, **25**:11–12, 1987.
36. J. S. B. Mitchell and E. L. Wynters. Watchman routes for multiple guards. In Proc. *Canad. Conf. Comput. Geom.*, pages 126–129, 1991.
37. J. S. B. Mitchell. Geometric shortest paths and network optimization. In *Handbook of Computational Geometry* (J.-R. Sack and J. Urrutia, editors). pages 633–701, Elsevier, 2000.
38. J. S. B. Mitchell and M. Sharir. New results on shortest paths in three dimensions. In Proc. *SCG*, pages 124–133, 2004.
39. B. J. Nilsson and D. Wood. Optimum watchmen routes in spiral polygons. In Proc. *Canad. Conf. Comput. Geom.*, pages 269–272, 1990.

40. B. J. Nilsson. Guarding art galleries; Methods for mobile guards. Ph.D. Thesis, Lund University, Sweden, 1995.
41. S. Ntafos. The robber route problem. *Inform. Process. Lett.*, **34**:59–63, 1990.
42. S. Ntafos. Watchman routes under limited visibility. *Comput. Geom.*, **1**:149–170, 1992.
43. S. Ntafos and L. Gewali. External watchman routes. *Visual Comput.*, **10**:474–483, 1994.
44. A. W. Roberts and V. D. Varberg. *Convex Functions*. Academic Press, New York, 1973.
45. R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, N.J., 1970.
46. M. Sharir and A. Schorr. On shortest paths in polyhedral spaces. *SIAM J. Comput.*, **15**:193–215, 1986.
47. D. Sunday. Algorithm 14: Tangents to and between polygons. See http://softsurfer.com/Archive/algorithm_0201/ (last visit: November 2008).
48. X. Tan, T. Hirata, and Y. Inagaki. An incremental algorithm for constructing shortest watchman route algorithms. *Int. J. Comp. Geom. and Appl.*, **3**:351–365, 1993.
49. X. Tan and T. Hirata. Constructing shortest watchman routes by divide-and-conquer. In Proc. *ISAAC*, LNCS 762, pages 68–77, Springer, Berlin / Heidelberg, 1993.
50. X. Tan and T. Hirata. Shortest safari routes in simple polygons. LNCS 834, pages 523–531, Springer, Berlin / Heidelberg, 1994.
51. X. Tan, T. Hirata, and Y. Inagaki. Corrigendum to ‘An incremental algorithm for constructing shortest watchman routes’. *Int. J. Comp. Geom. Appl.*, **9**:319–323, 1999.
52. X. Tan. Fast computation of shortest watchman routes in simple polygons. *Information Processing Letters*, **77**:27–33, 2001.
53. X. Tan. Shortest zookeeper routes in simple polygons. *Information Processing Letters*, **77**:23–26, 2001.
54. X. Tan. Approximation algorithms for the watchman route and zookeeper’s problems. In Proc. *Computing and Combinatorics*, LNCS 2108, pages 201–206, Springer, Berlin, 2001.
55. X. Tan and T. Hirata. Finding shortest safari routes in simple polygons. *Information Processing Letters*, **87**:179–186, 2003.
56. X. Tan. Approximation algorithms for the watchman route and zookeeper’s problems. *Discrete Applied Mathematics*, **136**:363–376, 2004.
57. X. Tan. Linear-time 2-approximation algorithm for the watchman route problem. In Proc. *Theory Applications Models Computation*, LNCS 3959, pages 181–191, Springer, Berlin, 2006.
58. D. S. Watkins. The QR Algorithm Revisited. *SIAM Review*, **(50)**1:133–145, 2008.
59. C.-K. Yap. Towards exact geometric computation. *Computational Geometry: Theory Applications.*, **7**:3–23, 1997.