A Solution to the Animal Problem

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Abstract

Let A be an animal. We consider magnifications of A in x-, y-, or z-direction. But this is not always possible because there may be voxels that block the magnification. To overcome this difficulty, we consider an AD-movement. This is performed by considering the magnification in two directions (e.g., y-direction and z-direction) such that any series of blocking voxels never forms a cycle. If a magnification is possible for two directions then it is also possible for the third direction. After all undilatable voxels have been erased, we can apply a previously shown result for showing that the animal problem is positively solvable.

This paper is a continuation of [4]. Let B be a well-composed picture [3] that does not contain any cavity or any tunnel. Is B SD-equivalent (i.e., equivalent by repeated simple deformations) to a single voxel? This problem was named *B-problem* by the late Azriel Rosenfeld. In [4], we have given a positive solution to this problem.

The animal problem proposed by Janos Pach was the question whether every animal can be reduced to a single unit cube by a finite sequence of moves, each consisting of either adding or deleting a cube provided the result is an animal again at each move. Azriel Rosenfeld called this problem the *A-problem*. In this animal problem, the deformation is not based on SD but must preserve animality of the picture at each step. This means that the B-problem is weaker than the A-problem.

In this paper, we give a positive solution to the animal problem. We assume that readers are familiar with basic definitions in digital topology, such as provided in [1, 2], and, in particular, with the discussion in [4].

An *animal* is defined as a topological 3-ball in $\mathbb{R}^3$, consisting of unit cubes (i.e., a subcomplex in the 3D grid cell space). In accordance with terms of digital topology, hereafter, we call a unit cube also a *voxel*. Let A be an animal consisting of black voxels (*black* means that a unit cube is present, i.e., in a given "object") and p be a black voxel.

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Definition 1 *An* additional animal voxel p *is defined by the property that* $A \cup \{p\}$ *is also an animal, and a* deletable animal voxel q *by the property that* $A \setminus \{q\}$ *is still an animal. An* animal voxel *is an additional or a deletable animal voxel.*

Usually, we denote a black voxel or a white voxel (note: *white* means that a unit cube is absent, i.e., in a "background component") by 1 or 0, respectively.

Furthermore, an animality-preserving deformation (abbreviated by AD) is defined as a finite sequence of deformations, each defined by either adding or deleting an animal voxel. Therefore, the animal problem is nothing but answering the following question:

Is an arbitrary animal A AD-equivalent to a single black voxel?

Let A be an animal. Then, the boundary of A must be a topological 2-sphere, so that it cannot contain any line (called a *singular line*) or any point (called a *singular point*) which destroys a topological 2-sphere. Figure 1 shows examples of singular lines and singular points.

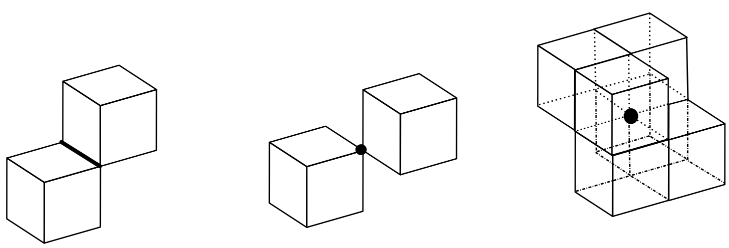

Figure 1: Singular line and singular point.

Notice: In Figure 1, we show a configuration of black voxels, but this is similar for white voxels that is obtained as the dual configuration. For example, the configurations in the middle and on the right are dual.

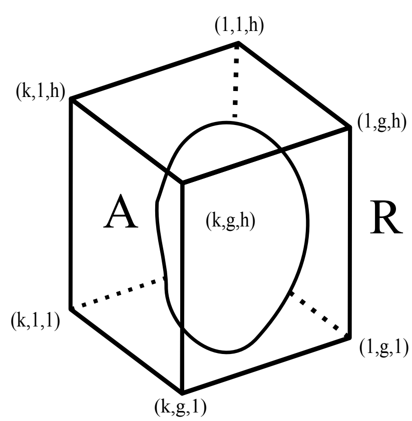

Figure 2: Animal A and the isothetic rectangular parallepiped R.

Let A be an animal. We consider the smallest isothetic rectangular parallelepiped (denoted by R) that contains A. See Figure 2 in which A is shown by a curve since its digital representation is complicated. In Figure 2, a voxel is put at each integer coordinate. Hereafter, we call the positive direction of x-axis, the positive direction of y-axis the south, the east, respectively.

Here, we consider scan order in an isothetic rectangular parallelepiped consisting of voxels.

(α) For horizontal planes, we scan from the top level to the bottom level.

(β) For a line (called x-line) that is parallel to x-axis on the same horizontal plane, we scan from east to west.

(γ) For voxels on the same x-line, we scan from south to north.

First, we do (γ) and then (β), and finally do (α). This scan order is denoted by (x-y-z). This may be called the reverse-lexicographic order. Notations such as (z-x-y), (z-y-x), $\cdots$ mean the similar scan orders. We start (x-y-z) from the south-east corner of the top level of R and end at the north-west corner of the bottom level of R. This procedure is called *one-pass scan* of R. If this one-pass scan is repeated, it is called *multi-pass scan* of R.

Let us consider a procedure to delete all animal voxels of an animal A. That is, if a voxel p is animal voxel of A, we delete p. To do this deletion systematically, we perform it in order of (x-y-z). In general, one-pass scan cannot delete all animal voxels of A, so that we use multi-pass scan. Many animals can be deformed to a single voxel by this multi-pass scan. However, there is an animal A such that it cannot be deformed to a single voxel by AD-deletion only. The Bing house is well-known as an example of such animals. In this note, we call this house the *standard Bing house*. Besides the standard Bing house, there are several animals which are not deformed to a single voxel in the procedure of animal voxel deletion only. In addition of the standard Bing house, we can construct various animals to which we can not apply the deletion of a voxel. There are three-storied (4-storied, $\cdots$, n-storied) Bing house; and also there is a small picture proposed by T. Shermer [5]. Such a picture is characterized by the property: It does not contain any arc and deleting of any voxel from the picture creates a tunnel (based on the adjacency (8, 26) of digital topology).

From this observation, it follows that we need mixed use of animal-preserving addition (abbreviated to AD-addition) and AD-deletion for the animal problem. But, this AD-addition is not always possible. See Figure 3.

By the way, “addition” can be regarded as “dilation of a black voxel” and “deletion” can be also regarded as “dilation of a white voxel”. In this situation, we use the term “dilation” for both cases. In Definition 2, we shall define a concept of “AD-undilatable pattern”. As mentioned above, an animal can contain AD-undilatable local patterns. To erase such a local pattern from an animal, we try to AD-dilate a voxel to the south, the east, and the upward direction, which are denoted by x-, y-, and z-, respectively.

Here, we give some definitions:

Definition 2 Let us consider a 4-connected set H of voxels on the same horizontal plane. Let H be the largest 4-connected set satisfying the following conditions.
(1) For every voxel p of H, the immediately upward voxel have the different color from that of p.
(2) For an arbitrary voxel p of H, the z-dilation of p creates always a singular line. Then, H is called a *z-AD-undilatable pattern.* Further, every voxel in a z-AD-undilatable pattern is called a *z-AD-undilatable voxel.*

We can similarly define an *x-AD-undilatable pattern*, a *y-AD-undilatable pattern*, and also an *x-AD-undilatable voxel*, and a *y-AD-undilatable voxel.* There are various (x-, y-, z-) AD-undilatable patterns. Some of them are illustrated in Figure 3.

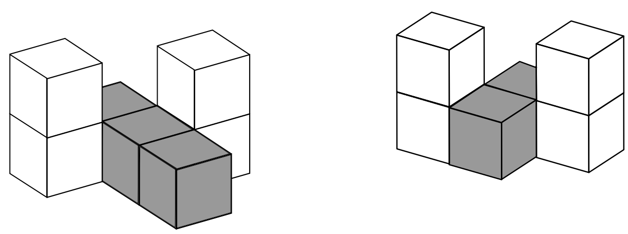

Figure 3: Examples of z-AD-undilatable patterns.

Definition 3 Let us consider a 4-connected set J of voxels on the same horizontal plane. J is not a z-AD-undilatable pattern, but if at least one z-AD-undilatable pattern is created by multi-pass z-AD-dilation of voxels of J, J is called a *z-AD-quasi-undilatable pattern.* Also, a voxel of a z-AD-quasi-undilatable pattern is called a *z-AD-quasi-undilatable voxel.*

Figure 4 illustrates some of z-AD-quasi-undilatable pattern. We can similarly define an *x-AD-quasi-undilatable pattern*, a *y-AD-quasi-undilatable pat-*

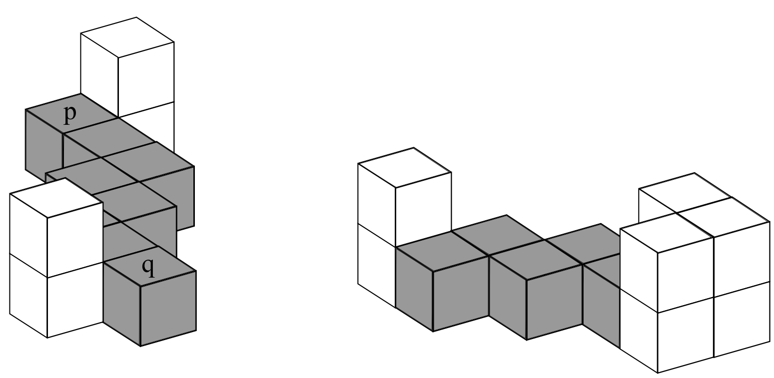

Figure 4: Examples of z-AD-quasi-undilatable patterns.

tern, and also an *x-AD-quasi-undilatable voxel*, and a *y-AD-quasi-undilatable voxel*.

As mentioned above, there are voxels that block z-dilation of z-AD-undilatable pattern by creating a singular line. A set of such blocking voxels of z-AD-undilatable pattern Π is called SIDE of Π. This notion is defined as follows:

Definition 4 Let Π be a z-AD-undilatble pattern. Let P be a set of voxels that have the same color as Π such that z-AD-dilation of a voxel of Π always creates a singular line as a common line with a voxel of P. Then, P is called *SIDE of* Π.

See Figure 5. Note that SIDE P has at least height 2.

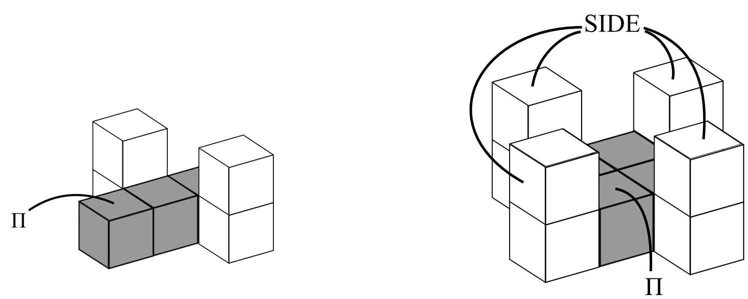

Figure 5: SIDE of Π.

Similarly, we can define SIDE of y-AD-undilatable pattern and x-AD-undilatable pattern.

As previously mentioned, we want to dilate R. This dilation is blocked by AD-undilatable pattern. Nevertheless, we consider dilations called *M-magnification* that is given later. Hereafter, the prefix such as x-, y-, z- is omitted, except special cases. For example, AD-undilatble pattern means one of x-AD-undilatable pattern, y-AD-undilatable pattern, and z-AD-undilatable pattern. Our dilation is not one-step dilation, but it is successively done as mentioned below. In this paper, we consider dilations for two-directions only (y-direction and z-direction). This is a key idea of this paper. Even if an animal is a 3D picture, but we treat it as a 2D picture of the "thickness".

By the way, voxels that block our successive dilations (will be called M-magnification, later) was SIDE, but there are another blocking voxel of these dilations. It is a voxel whose existence destroys animality. Such a voxel called a *stopping voxel*. See Figure 6.

If a voxel p can be M-magnified, p is called an *applicable voxel*, and if it is not possible, p is called a *non-applicable voxel*.

Thus, there are two sorts of non-applicable voxels for M-magnification: One is an undilatable voxel and another is a stopping voxel.

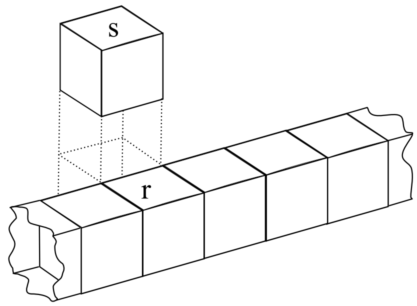

Figure 6: Stopping voxel (s is a stopping voxel of z-M-magnification of r).

Now, let us consider successive dilation of voxels in R. If there are no AD-undilatable patters (i.e., no AD-undilatable voxels), we can SD instead of AD. Therefore, our problem is solvable by making use of the result of [1]. Thus, we assume that at least one z-AD-undilatable voxel and also at least one y-AD-undilatable voxel.

Let the first picture on R be $R(\mathrm{O})$. Note that a given animal is in R. Then, we successively construct pictures $R(\mathrm{O}), R(\mathrm{I}), R(\mathrm{II}), \cdots$ as follows:

I. Construction of $R(\mathrm{I})$ from $R(\mathrm{O})$:

First, we construct $R(\mathrm{O}y)$ from $R(\mathrm{O})$ as follows: In $R(\mathrm{O})$, if $p(x, y, z)$ is black then we AD-dilate p until (x, My, z) and p is white then we AD-dilate until $(x, My + 1, z)$. Here, M is a sufficiently large integer such that $M > g + h$ and it is called the *magnification coefficient*. Also, this AD-dilation is called *y-M-magnification*. In this case, if there is a voxel that blocks this M-magnification, then we do nothing. Further, we do nothing for the case where some SIDE of a local pattern is created on a vertical (for y-axis) plane. (This stipulation is called SIDE-RULE.) For the case where a y-AD-undilatable pattern is created from a y-AD-quasi-undiltable pattern, this SIDE-RULE is not applied.

We apply successively this M-magnification to every vertical (that is orthogonal for y-axis) plane-by-plane, from the eastmost face of R to the westmost face of R. For each vertical plane, we do this in multi-pass of a raster scan. A picture obtained by the above deformation is denoted by $R(\mathrm{O}y)$. Note here that $R(\mathrm{O}y)$ is the smallest isothetic rectangular parallelepiped containing all of created black voxels. From $R(\mathrm{O}y)$, we construct $R(\mathrm{I})$ as follows:

In $R(\mathrm{O}y)$, if $p(x, y, z)$ is black then we AD-dilate p until (x, y, Mz) and p is white then we AD-dilate until $(x, y, Mz + 1)$. The M is the same as above. This AD-dilation is called *z-M-magnification*. In this case, if there is a voxel that blocks this M-magnification, then we do nothing. Further, we do nothing for the case where some SIDE is created on a horizontal plane (SIDE-RULE). We apply successively this M-magnification to every horizontal plane-by-plane, from the top level of R to the bottom level of R. A picture obtained by the above deformation is denoted by $R(\mathrm{I})$.

II. Construction of $R(\Lambda+1)$ from $R(\Lambda)$:

Hereafter, capital Greek letters mean the Roman numerals. By the similar method as we constructed $R(\mathrm{I})$ from $R(\mathrm{O})$, we define $R(\Lambda+1)$ from $R(\Lambda)$, inductively.

We aim at decreasing the number of AD-undilatable voxels as well as AD-quasi-undilatable voxels in $R(\mathrm{O})$ by constructing the sequence $R(\mathrm{O})$, $R(\mathrm{I}), \cdots$, $R(\Lambda), \cdots$. To this end, let us prove some claims:

Claim 1 Let p be a (white or black) voxel, and q be the next west voxel of p.
(1) If p and q have the same color and p is y-applicable, then q is also y-applicable.
(2) If p is y-non-applicable, then q is also y-non-applicable. (In this case, colors of p and q are irrelevant.)

(Proof) This is immediately known from the definitions of non-applicable voxels and applicable voxels. □

Let us consider a case where the colors of p, q are different. In this case, even if p is y-applicable, q is not always y-applicable. Because, p may be a y-AD-undilatable voxel so that q is not blocked by its SIDE. Now, assume that a (white or black) voxel p has been (z- or y-) AD-magnified. Then, we have a run (say, r) of magnified voxels of p. The head of this run is called the *head voxel of magnification of* p. Further, we consider a diagonal (digital) plane that intersects the horizontal plane by $\pi/4$. This diagonal plane is denoted by D, and the region above D is denoted by E. Also, the region below D is denoted by R^*. See Figure 7.

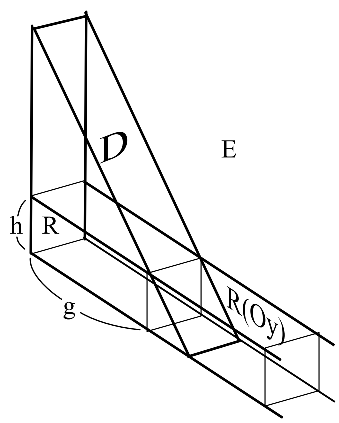

Figure 7: Positions of D, E, and R^*.

Claim 2 Assume that a (black or white) voxel t has been M-magnified. Let e be the head of magnified run of t. Then, e is in the outside of R^*.

(Proof) If t is in the outside of R^*, this is trivial. Even if t is in the inside of R^*, this is obvious from our definition of $M > g + h$. □

Claim 3 For any Λ, there is no non-applicable voxel in the outside of R^* in $R(\Lambda)$.

(Proof) For $R(\text{O})$, there is no non-applicable voxel in the outside of R^*. This is obvious, because the outside of R^* in $R(\text{O})$ are all white.

Let us consider $R(\text{O}y)$. E is the outside of R^*. There is no y-non-applicable voxel in E. The reason is as follows:

All black voxels in E are voxels created by y-M-magnification of each voxel. Thus, from the definition of M-magnification of black voxel it is known that there is no z-non-applicable voxel. Further, assume that there is a y-non-applicable voxel v in E. Let v be on a vertical plane V. Since we can take an integer N that is greater than M, V must be y-AD-dilated again by further one step. This is a contradiction with the assumption that v is y-non-applicable voxel. For a white voxel w, the above discussion is similar. Therefore, we know that there is no non-applicable voxel in E for $R(\text{O}y)$.

Next, we want to show that there is no non-applicable voxel in E for $R(\text{I})$. By making use of the similar discussion as above, we can prove this case. Anyway, we know that there is any non-applicable voxel in E for $R(\text{I})$.

Now, let us assume that this claim is true for a Λ. Then, we will show that the calim is also true for a $\Lambda+1$. But, this is shown by the quite similar argument as that we used in the case of $R(\text{I})$. Note that there is no non-applicable voxel in E for $R(\Lambda)$ — this is the hypothesis.

Therefore, by the induction on Λ we know that this claim is true. □

Claim 4 For any Λ, there is no AD-undilatable voxel in the outside of R in $R(\Lambda)$.

(Proof) This is provable by the quite similar method as in Claim 3. □

From Claim 3, we know that there is no non-applicable voxel in E. But, there may exist some non-applicable voxel (say, p) in the region R^*. Because there may exist an AD-undilatable voxel (say, q) in R such that q is a stopping voxel of p. Here, we introduce a notion of *subordination*. If a voxel r is a stopping voxel of s, we say that s is subordinate to r (denoted by $S(s, r)$. Further, if a voxel r is a stopping voxel of s and a voxel t is a stopping voxel of r, then we say again that s is subordinate to t. This subordinate relation is adequate and also it can be generalized. Thus, we have $S(s, r)$ and $S(r, t) \Rightarrow S(s, t)$. It is obvious that $S(p, p)$ is not true for an arbitrary voxel p.

Now, let us investigate the number of AD-undilatable voxels and AD-quasi-undilatable voxels in $R(\text{O})$, $R(\text{I})$, $R(\text{II})$, $\cdots$. Note here that there is no AD-undilatable voxel in the outside R for any $R(\Lambda)$ (where $\Lambda =$ O, I, II, III, $\cdots$).

Let us denote the total sum of the number of AD-undilatable voxels and the number of AD-quasi-undilatable voxels in $R(\Lambda)$ by $\#R(\Lambda)$.

Claim 5 Let $R(\Lambda)$ be a picture that contains at least one y-non-applicable and at least one z-non-applicable voxel. Then, there exists an integer Ξ such that $\#R(\Lambda) > \#R(\Xi)$.

Notice: If the assumption is not satisfied, we can easily erase all AD-undilatable voxels. This follows from the following observation:

Assume that there is no y-non-applicable in $R(\Lambda)$. From this assumption, we can always y-AD-dilate all voxels in $R(\Lambda)$, so that all z-non-applicable voxels are also AD-erased. Hence, this claim is trivially true.

(Proof) First of all, let us introduce a notion "AD-movement of SIDE". AD-movement of SIDE is performed as follows: After a SIDE P has been AD-magnified to a direction (e.g., y-direction), we AD-magnify again voxels (with different color from P) adjoining to P to the same y-direction. If this is possible, we say that SIDE P can be AD-moved.

Now, let us consider an AD-undilatable p in $R(\Lambda)$. Our assumption guarantees the existence of such a voxel p. Then, there must be SIDE that blocks AD-dilation of p. If this SIDE can be AD-moved, p may change into an applicable voxel. If so, this claim is true. But, there is a case where AD-movement of this SIDE needs again AD-movement of another SIDE. See Figure 8.

Under this situation, p changes into an applicable voxel after $R(\Lambda + 1)$.

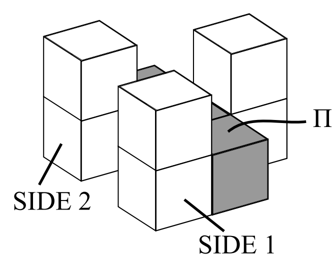

Figure 8: Two SIDEs of Π.

Even if SIDE 1 is y-AD-moved, a voxel p on z-AD-undilatable pattern Π is still a z-non-applicable voxel. When SIDE 2 is y-AD-moved, p may change into an applicable voxel.

However, there is a case where we need repeatedly AD-movement of SIDE. See Figure 9.

However, this process does not continue infinitely and also does not enter any cycle. This is known from the following consideration:

(1) We have considered AD-dilation to two-directions only (i.e., the east and the upward).

(2) AD-dilation of white voxel is blocked by a white voxel and AD-dilation of black voxel is blocked by a black voxel.

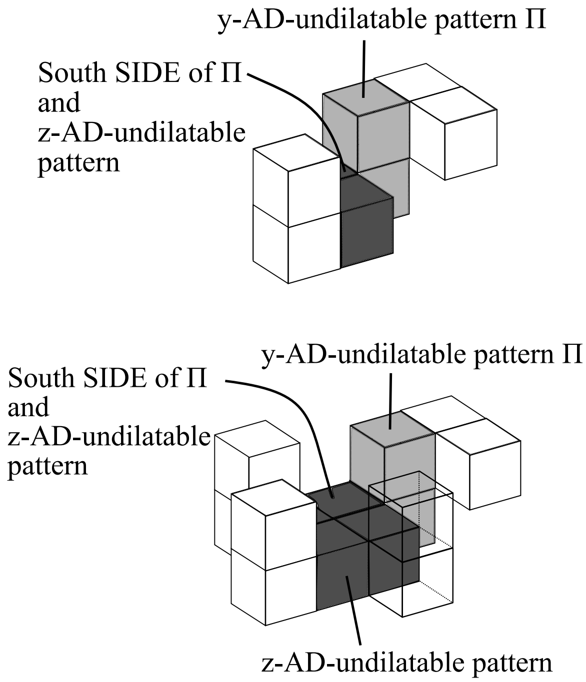

Figure 9: SIDE of SIDE.

(3) When we need M-magnification of either the south SIDE or the north SIDE, we take always the south SIDE. See Figure 10.

(4) Our picture is finite.

Therefore, we know that a non-applicable voxel change eventually into an applicable voxel (i.e., in some $R(\Xi)$, where $\Xi > \Lambda$).

But, the above consideration is not yet sufficient. Because, even if SIDE of a voxel p is erased, p may be subordinate to another voxel. In this case, let h be the first voxel of the subordinate relation containing p, i.e., h is no loger subordinated to another voxel. Then, p must be AD-undilatable voxel. Under such a circumstance, we begin the above discussion from h instead of p. Obviously, this does not continue infinitely.

However, our argument is not yet completed. M-magnification of a voxel u may create a new AD-undilatable pattern. But, this is avoided by our SIDE-RURE. Thus, we know that this claim is true. □

From Claim 5, we have the following proposition:

Proposition 1 Let A be a given animal. By making use of AD, we can erase all AD-undilatable voxels from A.

(Proof) This is easy. It is sufficient to repeat the discussion of the proof

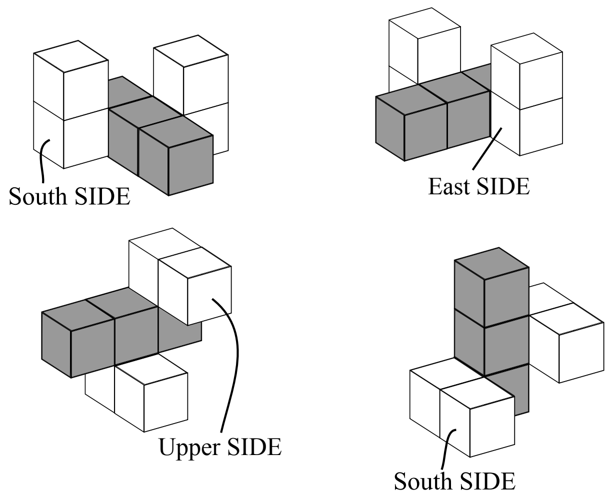

Figure 10: Adoption of SIDE.

of Claim 5. For a sufficient large Δ, there is $R(\Delta)$ that does not contain any AD-undilatable voxel. □

Note here that the obtained $R(\Delta)$ is not similar to the first animal A but it may be extremely large.

Theorem 1 Animal problem has a positive solution.

(Proof) $R(\Delta)$ of Proposition is obtained by AD (i.e., in animality-preserving) and it never contain any AD-undilatable pattern. Hence, we can use SD for the $R(\Delta)$.

Meanwhile, we have the result of [4] that gives a weak theorem of animal problem. At the present, we can use SD for $R(\Delta)$ instead of AD. Let Ω be a sufficiently large numeral such that the magnification method used in [4] can be applied to $R(\Omega)$. Then, from the result of [4], it is known that a given animal can be reduced into a single unit cube in AD (i.e., animality-preserving). □

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