The Topology of Incidence Pseudographs

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Abstract

Incidence pseudographs model a (reflexive and symmetric) incidence relation between sets of various dimensions, contained in a countable family. Work by Klaus Voss in 1993 suggested that this general discrete model allows to introduce a topology, and some authors have done some studies into this direction in the past (also using alternative discrete models such as, for example, abstract complexes or orders on sets of cells). This paper provides a comprehensive overview about the topology of incidence pseudographs. This topology has various applications, such as in modeling basic data in 2D or 3D digital picture analysis, or in describing polyhedral complexes. This paper addresses especially also partially open sets which occur, for example, in common (non-binary) picture analysis.

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1 Introduction

An incidence pseudograph $[S, I, \dim]$ models a (reflexive and symmetric) incidence relation I between sets c of dimension $\dim(c) \geq 0$, contained in a countable family S. (Relation I represents the symmetric completion of the *subset-of*-relationship.) This very general discrete model allows to introduce a topology, and to derive combinatorial formulas assuming some kind of regularity for the underlying geometry of *cells* $c \in S$. Obviously, the generality of this model allows for applications in a wide range of situations.

For example, digital (2D or 3D) pictures may be considered to be substructures of a regular orthogonal grid in (2D or 3D) space, and S would be a set of m-cells c (i.e., $\dim(c) = m$ with $0 \leq m \leq 3$) in this case; a pixel is a 2-cell, a voxel is a 3-cell, two pixels are *vertex-adjacent* if they are both incident with the same 0-cell, two voxels are *face-connected* iff they are both incident with the same 2-cell, and so forth.

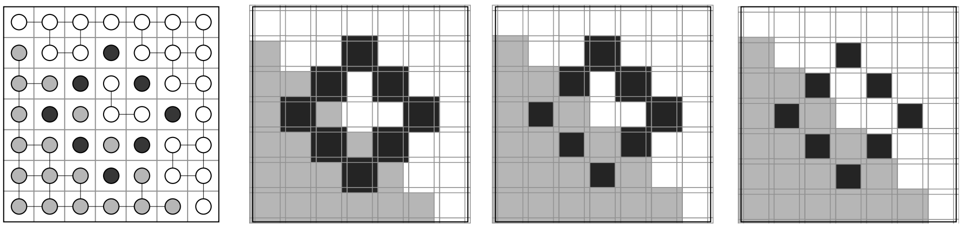

Fig. 1. Figure 5.21 in (7). The three-valued digital picture on the left is shown in three alternative topological interpretations. From left to right: black is 8-connected (forming a closed set), and gray is 4-connected (forming an open set), then gray is closed and white is open, and, finally, gray is again closed, but black is open.

See Figure 1 for 2-cells, 1-cells, and 0-cells of a 2D picture. The sketch in this figure indicates a partition of the digital plane into those cells, and in case of more than two values in a digital picture, one of those values defines non-open and non-closed regions, which will be studied as partially open in this paper. (For example, in the sketch on the right, black regions are open, there is one closed gray region, and one partially-open white region.

Definition 1 An *incidence structure* $[S, I, \dim]$ is defined by a countable set S of nodes, an *incidence relation* I on S that is reflexive and symmetric, and a function dim defined on S into a finite set $\{0, 1, \ldots, n\}$ of natural numbers.

Such a structure is called an *incidence pseudograph* (see Definition 2 below) if it satisfies additional constraints, such as having only finite sets $I(c)$ (i.e., being *locally finite*), or that a node $c' \in I(c)$ cannot be of the same dimensionality as node c. Incidence pseudographs allow us to model the topology of digital pictures, or of other discrete objects characterized by elements of varying dimensionality.

The book (7) decided for the model of incidence pseudographs for discussing the underlying digital topology of 2D or 3D digital pictures. Equivalently, also some model based on cells and their dimensionality (9), or on cells and their order (2) could have been used; however, graphs might be seen as an even more abstract model compared to families of cells.

Abstract complexes (9) are defined by cells of different dimensionality; see, for example, (5; 6; 8) for applications of this approach for defining fundamentals of binary image analysis. The equivalence between abstract complexes and incidence pseudographs was stated on page 223 in (7): Let $[S, I, \dim]$ be an incidence pseudograph. We define that $c < c'$ iff

$$c' \in I(c),\ c \neq c',\ \text{and}\ \dim(c) < \dim(c')$$

Let $c \leq c'$ iff $c < c'$ or $c = c'$. It follows that $[S, \leq, \dim]$ is an abstract complex. Note that the work in (5) (based on cells and their dimensionality) was mainly motivated by proving the correctness of a 3D surface scanning algorithm, which is also a central subject in (4), which defines and applies *digital spaces*, which are graph-theoretical models rather than cellular spaces.

Orders on sets of cells have been discussed in (2; 3), also for defining fundamentals of binary image analysis. (1) discussed the equivalence of orders on sets of cells with abstract complexes.

Incidence pseudographs have been introduced in (10) for discussing combinatorial properties of sets of pixels or voxels, considered to be grid points (note: not cells!) in 2D or 3D regular orthogonal grids. [Applying the topological discussion of (7) and what follows below, finite incidence pseudographs as considered in (10) are *open sets*.] The discussion of combinatorial properties (i.e., counts of nodes of various dimensions, and relations between such counts) has been complemented in (7) by also discussing closed sets. However, in this paper we will not discuss any of those combinatorial properties, and will focus on set-theoretical or topological properties instead. In this sense, this paper is not a review on incidence pseudographs in general by leaving one important subject fully out of our discussion.

This paper recalls the discussion of topological subjects of incidence pseudographs as given in (7) in a brief but concise form, and extends it then into a much more detailed analysis of topological properties of incidence pseudographs. In particular, this paper aims at presenting a topological concept for multi-valued (i.e., not just binary) pictures, having not just open or closed sets, but also *partially open* sets. Thus, this paper contains various new topological or set-theoretical results on incidence pseudographs, and the authors do not compare in every case what has been said already in (7) or not.

The paper is structured as follows: Section 2 introduces into incidence pseudographs. Sections 3 and 6 introduce the auxiliary notions of the rooted set and a descendence path, respectively. Section 4 introduces components and regions; subjects of major interest in this study. Section 5 then finally defines the topology by introducing open and closed sets. Section 7 shows that there is a unique topological closure for any finite set which has a connected nonempty core. Open, closed and complete sets are studied in Section 8. Section 9 shows that there is also a smallest open set containing a given set. Section 10 discusses a more technical concept (of 0-rooted sets), which is then applied in Section 11 for studying partially open sets and so-called 0-components and 0-regions. Section 12 concludes this paper.

2 Incidence Pseudographs

Let $G = [S, I, \dim]$ be an incidence structure. If n is the maximum of the range of dim, then we call G an *n-incidence structure* and say that $ind(G) = n$. A node $c \in S$ is called an *i-cell* if $\dim(c) = i$ and if $i = n$ we also say c is a *principal node* otherwise we say c is a *marginal node* of G. The set of all principal nodes of G is called the *core of* G, written $core(G)$, or $core(M)$ for $M \subseteq G$.

Two nodes $p, q \in S$ are *connected wrt* $M \subseteq S$ iff there exists a finite sequence $\{p_0, \ldots, p_n\}$ where

$p = p_0$ and $q = p_n$,
$(\forall i \in \{0, \ldots, n\}\ p_i \in M) \vee (\forall i \in \{0, \ldots, n\}\ p_i \in \overline{M})$, and
$\forall i \in \{0, \ldots, n-1\}\ p_i \in I(p_{i+1})$.

The sequence $\{p_0, \ldots, p_n\}$ is called a *path from* p *to* q. If also $p_i \in M$, for all $i \in \{0, \ldots, n\}$, we say that p and q are *connected in* M.

We say that p and q are *connected* if they are connected in S. A set $A \subseteq M \subseteq S$ is *connected wrt* M iff all $p, q \in A$ are connected wrt M. We say that A is *connected* if A is connected wrt S.

For $M \subseteq S$, the *complement* of M is defined as $\overline{M} = M \backslash S$. For $p \in \overline{M}$, the set $\{(p, q) : p$ and q are connected wrt $M\} \subseteq \overline{M}$ defines a *complementary component* of M.

Definition 2 An incidence structure $G = [S, I, \dim]$ is called an *incidence pseudograph* iff it has the following properties:

(1) For all $c \in S$, $I(c)$ is finite.
(2) The core of G is connected.
(3) Any finite set of principal nodes of G has at most one infinite complementary component of principal nodes.
(4) If $c' \in I(c), c' \neq c$, then $\dim(c) \neq \dim(c')$.
(5) Each marginal node of G is incident with at least one principal node of G.

G is said to be *monotonic* provided

(6) If $c' \in I(c), c'' \in I(c')$ and $\dim(c) \leq \dim(c') \leq \dim(c'')$ implies $c'' \in I(c)$.

Digital pictures, and subsets in those, are typically modeled by monotonic incidence pseudographs. However, those pseudographs allow us to describe discrete structures in a more general sense, and we also include non-monotonic pseudographs into our discussion (e.g., assume that *blocks* are either defined by bounded polyhedral objects, or a geometric arrangement of a finite number of blocks; incidence is only defined between polyhedral objects, or blocks of the same level of construction).

$G = [S, I, \dim]$ always denotes an incidence pseudograph in this paper; if no danger of confusion, a set S uniquely identifies "its" pseudograph G, and vice-versa. – For $i \in \mathbb{N}$ and $c \in S$, we define

$$\begin{aligned} I_i(c) &= \{c' \in I(c) : \dim(c') = i\} \\ G_i(c) &= \{c' \in I(c) : \dim(c') \geq i\} \\ G(c) &= \{c' \in I(c) : \dim(c') > \dim(c)\} \end{aligned}$$

We state four direct conclusions [with proofs].

If $i > j$, then $G_i(c) \subseteq G_j(c)$. [Assume $i > j$ and $c' \in G_i(c)$. Thus $c' \in I(c) \wedge \dim(c') \geq i$ which implies $c' \in I(c) \wedge \dim(c') \geq j$. Hence $c' \in G_j(c)$ and therefore $G_i(c) \subseteq G_j(c)$.]

If $i \leq ind(G)$, then $G_i(c) \neq \emptyset$. [Assume $i \leq ind(G)$ and let $c \in S$. There exists $p \in core(S) \cap I(c)$. Since $\dim(p) = ind(G) \geq i,\ p \in G_i(c)$.]

If $\dim(c) < ind(G)$, then $G(c) \neq \emptyset$. [Assume $\dim(c) < ind(G)$ and let $i = \dim(c)$. From Property (ii), we have $G(c) = G_{i+1} \neq \emptyset$.]

If $i = \dim(c)$, then $G(c) = G_{i+1}(c)$. [Assume $i = \dim(c)$ and note that $G(c) = \{c' \in I(c) : \dim(c') > \dim(c)\} = \{c' \in I(c) : \dim(c') \geq i + 1\} = G_{i+1}(c)$.]

The following was not yet defined this way in (7), and will prove to be useful. For $M \subseteq S$, $n = ind(G)$, and $0 \leq i \leq n$, we define M_i^+ recursively by

$$\begin{aligned} &M_n^+ = M \\ &M_{i-1}^+ = M_i^+ \cup \{c \in S : \dim(c) = i - 1 \wedge \emptyset \neq G(c) \subseteq M_i^+\} \end{aligned}$$

Finally, let $M^+ = M_0^+$. We say M^+ is the *completion* of M.

Definition 3 M is *complete* iff $M = M^+$.

If $core(M) = \emptyset$, then $M^+ = M$. [Suppose there exists a $c \in M^+ \backslash M$. Let $i = \dim(c)$ thus $c \in M_i^+ \backslash M_{i+1}^+$ and $\emptyset \neq G(c) \subseteq M_{i+1}$. There exists a principal node $p \in I(c)$. Note that $\dim(p) > i = \dim(c)$ and $p \in I(c)$ Therefore $p \in G(c) \subseteq M_{i+1}$ which implies that $p \in M^+$. Thus $p \in core(M)$ and hence $core(M) \neq \emptyset$.]

Lemma 4 *If* $n = ind(G)$ *and* $M \subseteq S$, *then*

(i) *For* $0 \leq i \leq j \leq n, M_i^+ \supseteq M_j^+$.
(ii) $M_0^+ = \bigcup_{i=0}^n M_i^+$
(iii) *If* $0 \leq i < n$, *then* $c \in M_i^+ \backslash M_{i+1}^+ \iff \dim(c) = i \wedge \emptyset \neq G(c) \subseteq M_{i+1}^+ \wedge c \notin M$.
(iv) *If* $i = \dim(c) \wedge c \in M^+ \backslash M$, *then* $c \in M_i^+ \wedge \emptyset \neq G(c) \subseteq M_{i+1}^+$.

PROOF. Property (i) follows immediately from the definition. Property (ii) follows from $0 \leq i \leq j \leq n, M_i^+ \supseteq M_j^+$. Property (iii) follows immediately from the definition.

To prove Property (iv), let $0 \leq i < n$ and assume $i = \dim(c)$ and $c \in M_0^+ \backslash M$ and let k be largest such that $c \in M_k^+$. Since $c \notin M = M_n^+$, we have $k < n$ and $c \in M_k^+ \backslash M_{k+1}^+$ and thus $k = i \wedge \emptyset \neq G_i(c) \subseteq M_{i+1}^+$. □

Theorem 5 *For $M \subseteq S$, M^+ is the smallest subset of S satisfying:*

(i) *$M \subseteq M^+$.*

(ii) *If $\emptyset \neq G(c) \subseteq M^+$, then $c \in M^+$.*

PROOF. Let $n = ind(G)$. Property (i) follows from the fact that $M = M_n^+ \subseteq M_0^+ = M^+$. To prove (ii), assume $\emptyset \neq G(c) \subseteq M^+$. If $c \in M = M_0^+$, then $c \in M^+$ so assume $c \notin M$. Let $i = \dim(c)$. Thus, by Lemma 4, $\emptyset \neq G(c) \subseteq M_{i+1}^+ \wedge c \in M_i^+ \backslash M_{i+1}^+$. Hence $c \in M^+$. Therefore M^+ satisfies Properties (i) and (ii).

Suppose C satisfies Properties (i) and (ii). Let $c \in M^+$. If $c \in M$, then, by Property (i), $c \in C$. Assume $c \in M^+ \backslash M$. Thus, by definition and Lemma 4, $c \in M_i^+ \backslash M_{i+1}$ where $i = \dim(c)$. Thus $c \in M_i^+ \backslash M$ where $i = \dim(c)$. We claim this is sufficient to insure $c \in C$.

Let $\mathbb{P}(i)$ be the statement "If $\dim(c) = i \wedge c \in M_i^+ \backslash M$, then $c \in C$". Let $n = ind(G)$. Since $M_n^+ = M$ and C satisfies Property (i), $\mathbb{P}(n)$ is true.

Assume $\mathbb{P}(j)$ is true for all j such that $i \leq j \leq n$ for some i such that $0 < i \leq n$ and let $\dim(c) = i - 1$ and $c \in M_{i-1}^+ \backslash M$. Thus $\emptyset \neq G(c) \subseteq M_i^+$. Let $c' \in G(c)$ and $k = \dim(c')$. Thus $c' \in M^+$ and $k \geq \dim(c) + 1 = i$. If $c' \in M$ then $c' \in C$ so assume $c' \notin M$. It follows that $c' \in M_k^+ \backslash M$. Since $i \leq k \leq n$, by assumption, $\mathbb{P}(k)$ is true and hence $c' \in C$. Thus $\emptyset \neq G(c) \subseteq C$. Since C satisfies Property (ii), $c \in C$. Therefore $M^+ \subseteq C$. □

Corollary 6 *M is complete iff $\emptyset \neq G(c) \subseteq M$ implies $c \in M$.*

PROOF. If M is complete then $M = M^+$ and, by Theorem 5, $\emptyset \neq G(c) \subseteq M$ implies $c \in M$. If $\emptyset \neq G(c) \subseteq M$ implies $c \in M$, then $M = M^+$ and hence is complete. □

3 Rooted Sets

A node $c \in M$ is said to be *rooted in M* iff c is incident to a principal node of M, otherwise c is said to be *unrooted in M*. *Rooted*(c) is the set of rooted nodes in M. *Unrooted*(c) is the set of unrooted nodes in M.

Definition 7 If $M = \mathit{Rooted}(M)$, M is said to be *rooted*.

From this we have the following:

$$\mathit{Rooted}(M) = \{c \in M : \mathit{core}(M) \cap I(c) \neq \emptyset\}$$

$$Unrooted(M) = M \backslash Rooted(M) = \{c \in M : core(M) \cap I(c) = \emptyset\}$$

If $M \neq \emptyset$ and M is rooted, then $core(M) \neq \emptyset$. [Assume $M \neq \emptyset$ and M is rooted and let $c \in M$. Since M is rooted there exists $p \in core(M) \cap I(c)$.]

The following two lemmas will be of repeated use later in this paper:

Lemma 8 *$Rooted(M)$ is rooted, and it is also complete if M is complete.*

PROOF. Let $R = Rooted(M)$. Note that $core(R) = core(M)$. Suppose $c \in R$. Thus $c \in M$ and $core(M) \cap I(c) \neq \emptyset$ and so $core(R) \cap I(c) \neq \emptyset$. Therefore R is rooted.

To show that R is complete, suppose there exists a $c \in R^+ \backslash R$. Since $R \subseteq M, R^+ \subseteq M^+$. Since M is complete we have $R^+ \subseteq M^+ = M$ and hence $c \in M \backslash R$. Thus $core(M) \cap I(c) = \emptyset$. Let $i = \dim(c)$. Since $c \in R^+ \backslash R$ we have $G(c) \subseteq R^+_{i+1}$. Since G is an incidence pseudograph, there exists a $p \in core(S) \cap I(c)$. Since M is complete and $c \in M$ we have $p \in M$. Hence $core(M) \cap I(c) \neq \emptyset$ which implies $c \in R$ but $c \notin R$. Therefore $R^+ = R$ and hence R is complete. $\square$

Lemma 9 *If $c \in M^+ \backslash M$ and $p \in core(S) \cap I(c)$, then $p \in M$.*

PROOF. Let $c \in M^+ \backslash M$ and $p \in core(S) \cap I(c)$. Let $i = \dim(c)$. Thus $c \in M^+_i \backslash M^+_{i+1}$ and $\emptyset \neq G(c) \subseteq M_{i+1}$. We have $p \in I(c)$ and $\dim(p) > \dim(c) = i$ so $p \in G(c)$ and therefore $p \in M_{i+1} \subseteq M^+$. Thus $p \in core(M^+) = core(M)$. Therefore $p \in M$. $\square$

This lemma allows the following

Corollary 10 (i) *If $c \in M^+ \backslash M$, then $core(M) \cap I(c) \neq \emptyset$*
(ii) *If M is rooted, then M^+ is rooted.*

PROOF. Property (i): Let $c \in M^+ \backslash M$. Since G is an incidence pseudograph, $\exists p \in core(S) \cap I(c)$. By Lemma 9 this implies $p \in M$ and since p is a principal node, that $p \in core(M)$. Hence $core(M) \cap I(c) \neq \emptyset$.

Property (ii): Assume M is rooted and let $c \in M^+$. If $c \in M$, then $core(M) \cap I(c) \neq \emptyset$. Otherwise, by Lemma 9, $core(M) \cap I(c) \neq \emptyset$. Therefore $M+$ is rooted. $\square$

4 Components and Regions

If M is complete and $C \subseteq M$, then C is called a *component* of M iff

(1) The principal nodes of C form a non-empty maximal connected (wrt M) subset of the principal nodes of M.

(2) If p is a principal node of $C, c \in M,$ and $c \in I(p)$, then $c \in C$.
(3) C is complete wrt G.

Definition 11 $M \subseteq S$ is said to be a *component* iff M is a component of M. A *region (of M)* is a finite component (of M).

If M is complete, rooted, and $core(M)$ connected and if C is a component of M, then $C = M$. [We have $C \subseteq M$. Let $c \in M$. We need to show $c \in C$. Since $core(M)$ is connected and $core(C)$ is a maximal connected subset of the $core(M)$ and since $core(M)$ is connected, we must have $core(C) = core(M)$. M is rooted so there exists a $p \in core(M) \cap I(c)$. Since $p \in core(C) \cap p \in I(c)$ and since C is a component, we have $c \in C$.]

By using Lemma 9, we can show the following

Corollary 12 *If C is a component of M, then $Rooted(C)$ is a rooted component of M.*

PROOF. Let $R = Rooted(C)$. Note $R \subseteq C \subseteq M$. The set $core(R) = core(C)$ is a nonempty maximal connected subset of $core(M)$ since C is a component of M.

If $c \in M, core(R) \cap I(c) \neq \emptyset$, then $core(C) \cap I(c) \neq \emptyset$ and $c \in M$. Therefore $c \in C$ since C is a component of M. Thus $c \in R$ since $c \in C$ and $core(C) \cap I(c) \neq \emptyset$.

To show R is complete, assume that there exists a $c \in R^+ \backslash R$. Since $R^+ \subseteq C^+ = C, c \in C$, and $c \notin R$ we have $core(C) \cap I(c) = \emptyset$. Since G is an incidence pseudograph, there exists a $p \in core(S) \cap I(c)$. By Lemma 9, $p \in C$. Therefore $p \in core(R) = core(C)$ and $p \in I(c)$ which contradicts $core(C) \cap I(c) = \emptyset$, since $c \notin C$. Therefore R is complete.

Let $c \in R$ which implies $c \in C$ and $core(C) \cap I(c) \neq \emptyset$. Therefore $core(R) \cap I(c) \neq \emptyset$ and so R is rooted. □

By using Lemma 9, we also show the following lemma, which will be used in the proof of the following theorem.

Lemma 13 *If M is complete and $p \in core(M)$, then M has a unique rooted component C containing p. Furthermore $C = core(C) \cup \{c \in M : I(c) \cap core(C) \neq \emptyset\}$.*

PROOF. Let $p \in core(M)$ for M complete. Let $A = \{c \in core(M) : c$ and p are connected wrt $M\}$. A is a nonempty maximal connected subset of $core(M)$.

Let $C = A \cup \{c \in M : A \cap I(c) \neq \emptyset\}$. $C \subseteq M$. Note that $core(C) = A$. Thus $core(C)$ is a non-empty maximal connected subset of M.

To show C is complete suppose there exists a $c \in C^+ \backslash C$. Since $c \notin C$, we have $A \cap I(c) = \emptyset$. There exists a principal node $q \in I(c)$. By Lemma 9, $q \in C$ which implies $core(M) \cap I(c) \neq \emptyset$. This implies that $c \in C$ which contradicts the assumption that $c \notin C$. Therefore $C = C^+$ and hence C is complete.

Let $q \in core(C), c \in M$, and $c \in I(q)$. Thus $c \in M$ and $A \cap I(c) \neq \emptyset$ which implies $c \in C$. Therefore C is a component of M containing p.

To show C is rooted let c be a marginal node of C. By the definition of C we have $core(C) \cap I(c) \neq \emptyset$ and hence C is rooted.

To show C is unique, assume R is a rooted component of M containing p. Since $core(R)$ and $core(C)$ are both maximal connected subsets of the $core(M)$ each containing p, we must have $core(C) = core(R) = A$. Let $c \in C$ and hence $c \in M$. If $c \in A$ then $c \in R$ so assume $c \notin A$ which implies $A \cap I(c) \neq \emptyset$ and hence $core(R) \cap I(c) \neq \emptyset$ which, since R is a component of M implies $c \in R$. So $C \subseteq R$. Let $c \in R$ which implies $c \in M$. Since R is rooted there exists a principal node $p \in I(c)$. Thus $A \cap I(c) \neq \emptyset$ which implies $c \in C$. Therefore $C = R$. $\square$

Theorem 14 (i) *If M is complete and rooted, then the rooted components of M form a partition of M.*
(ii) *If M is complete and not rooted, then the set consisting of $Unrooted(M)$ along with the rooted components of M is a partition of M.*

PROOF. Property (i): Following the previous lemma, for each $p \in core(M)$ let C_p be the unique rooted component of M containing p. Recall that $C_p = core(C_p) \cup \{c \in M : core(C_p) \cap I(c) \neq \emptyset\}$. Let $\mathbb{A} = \{C_p : p \in core(M)\}$. Let $p, q \in core(M)$ and assume $C_p \cap C_q \neq \emptyset$. Let $c \in C_p \cap C_q$. Since C_p and C_q are rooted, there exists a $p' \in core(C_p) \cap I(c)$ and there exists a $q' \in core(C_q) \cap I(c)$. We have p, p', c, q', q is a sequence of nodes in M each connected to the next and thus p and q are connected wrt M and thus $p \in C_q$ which implies $C_p = C_q$ since the rooted components of M containing p are unique. Thus $\mathbb{A}$ consists of disjoint subsets of M.

Let $c \in M$. Since M is rooted there exists a $p \in core(M) \cap I(c)$ and so $c \in C_p$ which implies $c \in \bigcup \mathbb{A}$. Since $\bigcup \mathbb{A} \subseteq M$ we conclude that $M = \bigcup \mathbb{A}$.

Property (ii): At first we show that, if C is a component of $Rooted(M)$, then C is a rooted component of M. – Let $R = Rooted(M)$ and let C be a component of R. Note that $core(R) = core(M)$ and thus $core(C)$ is a maximal connected subset of $core(M)$.

Assume $p \in core(C)$, $c \in M$, and $c \in I(p)$. Since $core(C) \subseteq core(R)$, $p \in core(R), c \in M$ and $c \in I(p)$ which implies that c is rooted in M and hence $c \in R$. Since C is a component of R we have $c \in C$, and since C, being a component of R is complete, we have C is a component of M. Since $C \subseteq Rooted(M)$, we have $core(M) \cap I(c) \neq \emptyset$, for all $c \in C$. Therefore C is a rooted component of M.

Now let $K = M \backslash Unrooted(M) = M \cap Rooted(M) = Rooted(M)$. By Lemma 8, K is complete and rooted. Thus, by Property (i), K is partitioned by the rooted components of K. Let $\mathbb{P}$ be the collection of the rooted components of K along with the set $Unrooted(M)$. As shown above, we also have that the components of K are rooted components of M. Clearly $M = \bigcup \mathbb{P}$. We have the rooted components of M are disjoint and disjoint from $Unrooted(M)$. Therefore $\mathbb{P}$ partitions M. $\Box$

5 Definition of Topology; Closed and Open Sets

Definition 15 $M \subseteq S$ is said to be *closed* iff, for all $c \in M$ and for all $c' \in I(c)$ with $\dim(c') < \dim(c)$, it follows that $c' \in M$. M is said to be *open* iff $\overline{M} = S \backslash M$ is closed.

As usual, the family of all open sets defines a *topology*, here on the given incidence pseudograph. A set M is closed iff $\overline{M}$ is open. [Because of $\overline{\overline{M}} = M$.]

A node c of a set M is called an *inner node of* M iff $I(c) \subseteq M$, otherwise it is called a *border node of* M.

Definition 16 M^{∇} is the set of inner nodes of M. δM is the set of border nodes of M and is called the *border of* M.

Theorem 17 *If M is closed, then both M and M^{∇} are complete.*

PROOF. Set M: Suppose M is closed. We claim $M_i^+ \subseteq M$ for $0 \leq i \leq n$ where $n = ind(G)$. Recall $M_n^+ = M$ so the claim is true for $i = n$.

Assume $M_i^+ \subseteq M$ for some $0 < i \leq n$ and let $c \in M_{i-1}^+$. Thus $c \in M_i^+$ (and hence $c \in M$) or that $\dim(c) = i - 1 < n$ and $\emptyset \neq G(c) \subseteq M_i^+$. Assume $c' \in G(c) \subseteq M_i^+$. Thus $c' \in I(c)$ and $\dim(c') > \dim(c)$. By assumption $M_i^+ \subseteq M$. Hence $c' \in M$. Since M is closed, this implies $c \in M$. Therefore $M_i^+ \subseteq M$ for all i satisfying $0 < i \leq n$. This implies $M^+ = M$ and therefore M is complete.

Set M^{∇}: Assume M is closed. We claim $(M^{\nabla})_i^+ \subseteq M^{\nabla}$ for $0 \leq i \leq n$.

Assume $(M^{\nabla})_i^+ \subseteq M$ for some $0 < i \leq n$ and let $c \in (M^{\nabla})_{i-1}^+$. Thus $c \in (M^{\nabla})_i^+$ (and hence $c \in M^{\nabla}$) or that $\dim(c) = i - 1 < n$ and $\emptyset \neq G(c) \subseteq (M^{\nabla})_i^+$. Assume $c' \in G(c) \subseteq (M^{\nabla})_i^+$. Thus $c' \in I(c)$, $\dim(c') > \dim(c)$, and $c' \in (M^{\nabla})_i$ which, by assumption, implies $c' \in M^{\nabla}$. Since $c \in I(c')$ this implies that $c \in M$.

To show $c \in (M^{\nabla})$ let $b \in I(c)$. If $dim(b) > dim(c)$, then, since $G(c) \subseteq M^{\nabla}$, we have $b \in M^{\nabla}$ and hence $b \in M$. If $\dim(b) = \dim(c)$, then $b = c$ which implies $b \in M$. If $\dim(b) < \dim(c)$, then $b \in M$ since M is closed. This implies $(M^{\nabla})^{+} = M^{\nabla}$ and therefore M^{∇} is complete. $\square$

We also note that M is both open and closed iff its border is the empty set (i.e... $\delta M = \emptyset$ iff $M = M^{\nabla}$). [Obviously, $\delta M = M \backslash M^{\nabla}$ and thus $\delta M = \emptyset$ iff $M = M^{\nabla}$. – Now assume that M is open and closed and let $c \in M$. To show $c \in M^{\nabla}$, let $c' \in I(c)$. If $\dim(c') > \dim(c)$, then $c' \in M$ since M is open. If $\dim(c') = \dim(c)$, then $c' = c$ and hence $c \in M$. If $\dim(c') < \dim(c)$, then $c' = c$ and hence $c \in M$ since M is closed. Therefore $M = M^{\nabla}$. – On the other hand, assume that $M = M^{\nabla}$ and suppose $c \in M$ satisfies $c' \in I(c)$. It follows that $I(c) \subseteq M$ since $M = M^{\nabla}$ and thus $c' \in M$.]

The following examples illustrate various situations which may occur.

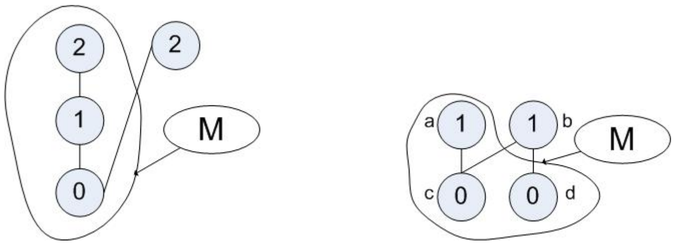

Fig. 2. Left: A finite, closed (and hence complete), non-empty M which has a non-rooted component. Right: An M which is closed (and hence complete) with $M \neq core(M)^{+}$.

Figure 2 shows that there exists a finite, closed (and hence complete), non-empty M which has a non-rooted component (left), and also (right) that there exists an M which is closed (and hence complete) with $M \neq core(M)^{+}$; for this, let G be defined by the diagram; let M={a,c,d}. M is closed but $core(M)^{+} = \{a, c\} \neq M$. Note that these pseudographs are not monotonic.

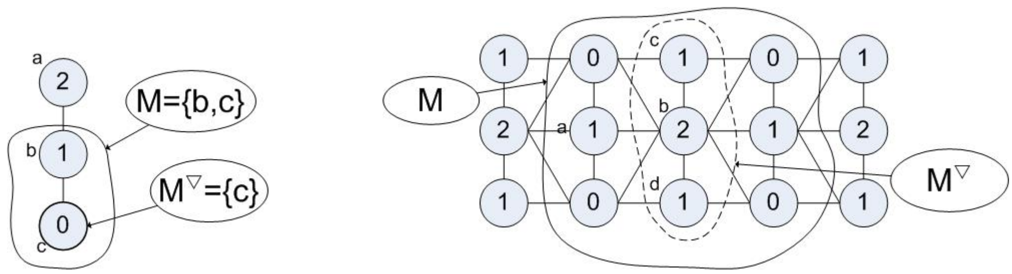

Fig. 3. Left: An M which is closed (and hence complete) and M^{∇} not open. Right: An M which is closed, not open, and M^{∇} not closed.

Figure 3 shows on the left that there exists an M which is closed (and hence complete) and M^{∇} not open; for this let S and M be defined by the diagram; $M = \{b, c\} = M^{+}$, $M^{\nabla} = \{c\}$, and $\overline{M^{\nabla}} = \{a, b\}$ which is not closed since it is missing c. Therefore M^{∇} is not open. The figure shows on the right that there exists an M which is closed, not open, and M^{∇} not closed. For this, let S and

M be defined by the diagram; $M^{\nabla} = \{b, c, d\}$, $b \in M^{\nabla}$, $\dim(a) < \dim(b)$, and $a \notin M^{\nabla}$. Therefore M^{∇} is not closed.

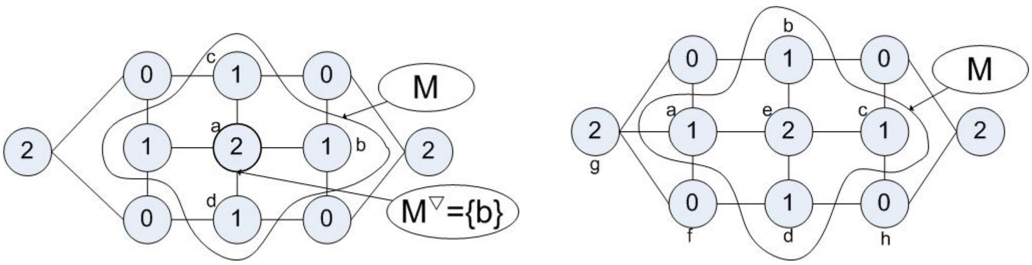

Fig. 4. Left: An M which is complete, open, not closed, and M^{∇} is not complete. Right: An M which is complete and δM is not complete, not closed, and not open.

Figure 4 shows on the left that there exists an M which is complete, open, not closed, and M^{∇} is not complete. For this, let S and M be defined by the diagram; $M^{\nabla} = \{a\}$. Note $\emptyset \neq \{c' \in I(b) : \dim(c') > \dim(b)\} = \{a\} \subseteq M^{\nabla}$, but $b \notin M^{\nabla}$. – The figure shows on the right that there exists an M which is complete and δM is not complete, not closed, and not open. For this, let S and M be defined by the diagram; $\delta M = \{a, b, c, d\}$. Note $\emptyset \neq \{c' \in I(h) : \dim(c') > \dim(h)\} = \{c, d\} \subseteq \delta M$. But $h \notin \delta M$. Therefore δM is not complete and hence not closed. Furthermore, $g \in \overline{\delta M}$, $\dim(g) > \dim(a)$, and $a \notin \overline{\delta M}$. Therefore δM is not open.

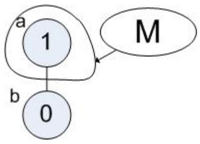

Fig. 5. An M which is open but not complete.

Figure 5 shows that there exists an M which is open but not complete. For this, let $S = \{a, b\}$ and $M = \{a\}$ as defined by the diagram. To show M is open note that $\overline{M} = \{b\}$ which is closed since $dim(b) = 0$ and thus there is no node $c \in I(b)$ such that $dim(c) < dim(b)$. To show M is not complete we note that $\emptyset \neq G(b) \subseteq M$ which implies $b \in M^{+} \setminus M$.

Lemma 18 *M is open iff, for all $c \in M$ and $c' \in I(c)$ with $\dim(c') > \dim(c)$, it follows that $c' \in M$.*

PROOF. Assume M is open, $c \in M$, $c' \in I(c)$ and $\dim(c') > \dim(c)$. If $c' \in \overline{M}$ which is closed since M is open, we would have $c \in \overline{M}$. Therefore $c' \in M$.

Assume that for all $c \in M$ and $c' \in I(c)$ with $\dim(c') > \dim(c)$ it follows that $c' \in M$, and suppose $c \in \overline{M}$, $c' \in I(c)$ and $\dim(c') < \dim(c)$. If $c' \in M$ this would imply that $c \in M$. Thus $c' \in \overline{M}$. Therefore $\overline{M}$ is closed and hence M is open. $\square$

6 Descendence Paths

This section prepares for important considerations in the following section by providing and discussing the notion of a descendence path.

A sequence of nodes $\{p_0, \ldots, p_k\}$ is called a *descendence path (from p_0 to p_k)* iff, for all $i \in \{0, \ldots, k-1\}$, $\dim(p_{i+1}) > \dim(p_i)$ and $p_{i+1} \in I(p_i)$. For example, in a 3D regular grid, we may start with a grid vertex p_0, continue with a grid edge p_1 which is incident with this vertex, then with a grid face p_2 incident with this edge, and finally a grid cube p_3 incident with this face.

A descendence path $\{p_0, \ldots, p_k\}$ is called a *descendence path wrt M (from p_0 to p_k)* iff for $0 \leq i < k$, $p_i \notin M$ and $p_k \in S$. Note that for any node c, $\{c\}$ is a descendence path (wrt any M) from c to c.

For $M \subseteq S, i \in \mathbb{N}$ define

$$\mathbb{C}(M, i) = \{c \in S : \exists \text{ descendence path } \{p_0, \ldots, p_i\} \text{ with } c = p_0 \wedge p_i \in M\}$$
$$M^\bullet = \cup_{i=0}^{\infty} \mathbb{C}(M, i)$$

Note that $M = \mathbb{C}(M, 0)$, so $M \subseteq M^\bullet$. Also note that for $n = ind(G)$, $\quad M^\bullet = \cup_{i=0}^{n} \mathbb{C}(M, i)$ since $\mathbb{C}(M, i) = \emptyset$ for $i > n$. We say c' is a *descendent of c* iff there exists a descendence path from $c = p_0$ to $c' = p_k$. Let $D(c) = \{c' :$ c' is a descendent of $c\}$ and $D^M(c) = \{c' : \exists$ descendence path wrt M from c to c' $\}$.

If $c' \in I(c) \wedge \dim(c') > \dim(c)$, then $D(c') \subseteq D(c)$. [Let $b \in D(c')$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ from c' to b. Then $\{c, p_0, \ldots, p_k\}$ is a descendence path from c to b. Thus $b \in D(c)$. Therefore $D(c') \subseteq D(c)$.]

We have that $D(c) = \{c\} \cup \bigcup\{D(c') : c' \in I(c) \wedge \dim(c') > \dim(c)\}$. [Let $b \in D(c)$ and $b \neq b$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ from c to b. Note that $b \in D(p_1), p_1 \in I(c), \dim(p_1) > \dim(c)$ since $c = p_0$. Therefore $D(c) \subseteq \{c\} \cup \bigcup\{D(c') : c' \in I(c) \wedge \dim(c') > \dim(c)\}$. – Let $b \in \{c\} \cup \bigcup\{D(c') : c' \in I(c) \wedge \dim(c') > \dim(c)\}$. If $b = c$, then $b \in D(C)$ so assume $b \neq c$. Then there exists $c' \in I(c), \dim(c') > \dim(c)$, and $b \in D(c')$. Thus there is a descendence path $\{p_0, \ldots, p_k\}$ from c' to b. Define $s_0 = c$ and $s_i = p_{i-1}$ for $1 \leq i \leq k+1$. Then $\{s_0, \ldots, s_{k+1}\}$ is a descendence path from c to b and hence $b \in D(c)$.]

For $0 \leq i \leq ind(G)$, we define $D_i(c) = \{c' \in D(c) : \dim(c) = i\}$ and $D_i^M(c) = \{c' \in D^M(c) : \dim(c) = i\}$.

If $A = M \cup \{c \in S \backslash M : D_n(c) \subseteq M\}$, $n = ind(G)$, and if $\{p_0, \ldots, p_k\}$ is a descendence path for which there exists a $p_i \in A \backslash M$, then $p_j \in A$, for all j, with $i \leq j \leq k$. [Assume $\{p_0, \ldots, p_k\}$ is a descendence path such that there exists a $p_i \in A \backslash M$. Let $i \leq j \leq k$. We need to show $D_n(p_j) \subseteq M$. If $j = i$ then we have $p_j \in A \backslash M$ and so $D_n(p_j) \subseteq M$. Assume $i < j$. Let $s \in D_n(p_j)$ Thus there exists a descendence path $\{s_0, \ldots, s_m\}$ from p_j to s, with $\dim(s) = n$. For $0 \leq q \leq m - i + j$, define

$$t_q = \begin{cases} p_{i+q} & 0 \leq q \leq j - i \\ s_{q+i-j} & j - i < q \leq m - i + j \end{cases}$$

Then $\{t_0, \ldots, t_{m-i+j}\}$ is a descendence path from p_i to s and hence $s \in D_n(p_i) \subseteq M$. Therefore $D_n(p_j) \subseteq M$ and thus $p_j \in A$.]

If $c \notin core(S) \wedge D^M(c) \subseteq M$, then $c \in M^+$. [Assume $c \notin core(S) \wedge D^M(c) \subseteq M$ and suppose $c \notin M^+$. Let $i = \dim(c)$. Since $c \notin M^+, c \notin M_i$. Thus $\emptyset = G(c)$ or $G(c) \not\subseteq M_{i+1}$. Since $c \notin core(S)$ we have $G(c) \neq \emptyset$. Thus there exists $c' \in G(c) \backslash M_{i+1}$. So $c' \in I(c), \dim(c') \geq i + 1 > \dim(c)$. $c' \notin M_{i+1}$ implies $c' \notin M$. Let $p_0 = c, p_1 = c'$, then $\{p_0, p_1\}$ is a descendence path from c to $c', p_i \notin M$ for $0 \leq i < 1$ Therefore $c' \in D^M(c) \backslash M$. This implies $D^M(c) \not\subseteq M$. Therefore $c \in M^+$.]

Note that, if $p_i \notin M$ for all i satisfying $0 \leq i \leq k$, then $p_k \in D^M(c)$. Thus it follows that if a node c, which is in the completion of a set M, had a descendent b which is not in the completion of M, then every descendence path from c to b must contain at least one member of M. We restate this fact in the following:

Corollary 19 *If $c \in M^+$ and $\{p_0, \ldots, p_k\}$ is a descendence path with $c = p_0$ and $p_k \notin M^+$, then there exists i, $0 \leq i \leq k$, with $p_i \in M$.*

7 Topological Closure

In this section we show that any set $M \subseteq S$ does have a unique "topological closure".

Lemma 20 *If D is closed and $M \subseteq D$, then*

(i) *for all $i \in \mathbb{N}$, $\mathbb{C}(M, i) \subseteq D$*
(ii) *$M^\bullet \subseteq D$*

PROOF. Property (ii) follows from Property (i) since $M^\bullet = \bigcup_{i=0}^{\infty} \mathbb{C}(M, i)$.

Let $n = ind(G)$. Note that $\mathbb{C}(M, i) = \emptyset$, for all $i > n$. We prove Property (i) by induction. Since $\mathbb{C}(M, 0) = M$ we have $\mathbb{C}(M, 0) \subseteq D$.

Assume $\mathbb{C}(M, i) \subseteq D$ and let $c \in \mathbb{C}(M, i+1)$ for some $0 \leq i < n$. By definition there exists a descendence path $\{p_0, \ldots, p_{i+1}\}$ with $c = p_0$ and $p_{i+1} \in M$. For $0 \leq j \leq i$ let $s_j = p_{j+1}$. We have $s_i = p_{i+1} \in M$ so $s_0 \in \mathbb{C}(M, i)$. By the assumption we have $s_0 \in D$. Hence $p_1 \in D$ and $p_0 \in I(p_1)$. Since D is closed, $c = p_0 \in D$. $\square$

Corollary 21 *If D is closed, $M \subseteq D$, and if $\{p_0, \ldots, p_k\}$ is a descendence path with $p_k \in M$, then $p_0 \in D$.*

Theorem 22 *$M^\bullet$ is the smallest closed set containing M.*

PROOF. $M = \mathbb{C}(M, 0) \subseteq M^\bullet$. To show $M^\bullet$ is closed, let $c \in M^\bullet$ and $c' \in I(c)$ such that $\dim(c') < \dim(c)$. Then there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $c = p_0$ and $p_k \in M$. Define $s_0 = c'$ and for $1 \leq j \leq k+1$, define $s_j = p_{j-1}$. Thus $\{s_0, \ldots, s_{k+1}\}$ is a descendence path with $c' = s_0 \wedge s_{k+1} \in M$. Hence $c' \in M^\bullet$ and therefore $M^\bullet$ is closed.

Suppose $M \subseteq D$ and D is closed. By Lemma 20, $M^\bullet \subseteq D$. $\square$

This theorem now allows us to formulate the following important

Definition 23 Let $M \subseteq S$; we denote the unique (topological) *closure* of M by $M^\bullet$.

Corollary 24 $M^\bullet = (M^+)^\bullet$

PROOF. $M^+ \subseteq M^\bullet$ and $M^\bullet$ is closed. It follows from Theorem 22 that $(M^+)^\bullet \subseteq M^\bullet$. Since $M \subseteq M^+$, it follows that $M^\bullet \subseteq (M^+)^\bullet$. Therefore $M^\bullet = (M^+)^\bullet$. $\square$

Corollary 25 *If M is finite, then $M^\bullet$ is finite.*

PROOF. Assume M is finite. Thus $\mathbb{C}(M, 0) = M$ is finite. Let $n = ind(G)$ and assume $\mathbb{C}(M, i)$ is finite for $0 \leq i < n$. We show that

$$\mathbb{C}(M, i+1) = \{c \in S : \exists c' \in \mathbb{C}(M, i) \cap I(c) \wedge \dim(c) < \dim(c')\}$$

Let $A = \{c \in S : \exists c' \in \mathbb{C}(M, i) \cap I(c) \text{ with } \dim(c) < \dim(c')\}$ and let $c \in \mathbb{C}(M, i+1)$. By definition, there exists a descendence path $\{p_0, \ldots, p_{i+1}\}$ with $c = p_0$ and $p_{i+1} \in M$.

Let $c' = p_1$. From the definition of a descendence path we have $c = p_0 \in I(c')$ and $\dim(c) < \dim(c')$. For $0 \leq j \leq i$ let $s_j = p_{j+1}$. Then, $\{s_0, \ldots, s_i\}$ is a descendence path with $c' = s_0$ and $s_i = p_{i+1} \in M$. Hence $c' \in \mathbb{C}(M, i)$ and thus $c \in A$.

Let $c \in A$. Then there exists a $c' \in \mathbb{C}(M, i)$ such that $c' \in I(c)$ and $\dim(c) < \dim(c')$. There exists a descendence path $\{p_0, \ldots, p_i\}$ such that $c' = p_0$ and $p_i \in M$. Let $s_0 = c$ and for $1 \leq j \leq i+1$, let $s_j = p_{j-1}$. We note that $\{s_0, \ldots, s_{i+1}\}$ is a descendence path from C to $p_i \in M$ and hence $c \in \mathbb{C}(M, i+1)$. Therefore $\mathbb{C}(M, i+1) = A$.

Note that if $c \in \mathbb{C}(M, i{+}1)$, then there exists a $c' \in \mathbb{C}(M, i)$ such that $c \in I(c')$. And thus $c \in \bigcup\{I(c') : c' \in \mathbb{C}(M, i)\}$. Therefore

$$\mathbb{C}(M, i+1) \subseteq \bigcup\{I(c') : c' \in \mathbb{C}(M, i)\}$$

which is a finite union of finite sets and therefore finite. Thus $M^\bullet = \cup_{i=0}^{n} \mathbb{C}(M, i)$ is also finite. □

Corollary 26 *If M is finite, then M^+ is finite.*

PROOF. $M^+ \subseteq M^\bullet$ and the previous corollary. □

Lemma 27 *If G is monotonic and $M \subseteq S$, then $M^\bullet = M \cup \mathbb{C}(M, 1)$.*

PROOF. Let $A = M \cup \mathbb{C}(M, 1) = \mathbb{C}(M, 0) \cup \mathbb{C}(M, 1)$. Let $n = \mathit{ind}(G)$. We will show $\mathbb{C}(M, i) \subseteq \mathbb{C}(M, 1)$ for all i, $1 \leq i \leq n$, by induction. Clearly it is true for $i = 1$.

Assume $\mathbb{C}(M, i) \subseteq \mathbb{C}(M, 1)$ and let $c \in \mathbb{C}(M, i+1)$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $c = p_0$ and $p_{i+1} \in M$. For $0 \leq j \leq i$ define $s_j = p_{j+1}$. So $p_1 = s_0 \in \mathbb{C}(M, i)$. By assumption, $p_1 = c_0 \in \mathbb{C}(M, 1)$. Thus there exists a descendence path $\{t_0, t_1\}$ such that $p_1 = t_0$ and $t_1 \in M$. We have $c = p_0 \in I(p_1)$, $\dim(c) < \dim(p_1)$, $t_0 = p_0 \in I(t_1)$, and $\dim(p_1) < \dim(t_1)$. Since G is monotonic this implies $c \in I(t_1)$. Thus, for $r_0 = c, r_1 = t_1$, $\{r_0, r_1\}$ is a descendence path with $c = r_0$ and $r_1 \in M$. Therefore $c \in \mathbb{C}(M, 1)$. □

Corollary 28 *If $G = [S, I, \dim]$ is monotonic and if M is a rooted subset of S, then $M^\bullet$ is rooted.*

PROOF. Let c be a marginal node of $M^\bullet$. If $c \in M$, then $\mathit{core}(M) \cap I(c) \neq \emptyset$ since M is rooted. So assume $c \notin M$. Since G is monotonic we have from Lemma 27 that $c \in \mathbb{C}(M, 1)$. So there exists a descendence path $\{p_0, p_1\}$ such that $c = p_0$ and $p_1 \in M$. If $p_1 \in \mathit{core}(M) = \mathit{core}(M^\bullet)$, then $p_1 \in \mathit{core}(M^\bullet) \cap I(c)$. Assume $p_1 \notin \mathit{core}(M)$. Thus p_1 is marginal in M. Since M is rooted there exists a $p \in \mathit{core}(M) \cap I(p_1)$. We have $c \in I(p_1)$, $\dim(c) < \dim(p_1)$, $p_1 \in I(p)$ and $\dim(p_1) < \dim(p)$. Since G is monotonic this implies $c \in I(p)$. Thus $p \in \mathit{core}(M) = \mathit{core}(M^\bullet)$. Therefore $M^\bullet$ is rooted. □

We also state the following three set-theoretical relations:

(i) $\mathit{core}(M^+) = \mathit{core}(M)$: Let $n = \mathit{ind}(G)$ and let $p \in \mathit{core}(M^+)$. Thus $\dim(p) = n$ which implies $p \in M_n^+ = M$ which implies $p \in \mathit{core}(M)$. Therefore $\mathit{core}(M^+) \subseteq \mathit{core}(M)$. Since $m \subseteq M^+$, we have $\mathit{core}(M) \subseteq \mathit{core}(M^+)$. Therefore $\mathit{core}(M^+) = \mathit{core}(M)$.

(ii) $\mathit{core}(M^\bullet) = \mathit{core}(M)$: Let $n = \mathit{ind}(G)$. Since $M \subseteq M^\bullet$, we have $\mathit{core}(M) \subseteq \mathit{core}(M^\bullet)$. Let $p \in \mathit{core}(M^\bullet)$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $p = p_0$ and $p_k \in m$. Since $\dim(p) = n, k = 0$. Thus $p = p_k \in M$. Therefore $\mathit{core}(M^\bullet) = \mathit{core}(M)$.

(iii) $M^+ \subseteq M^\bullet$: $M^\bullet$ is closed and therefore complete and contains M. Since M^+ is the smallest complete set containing M, $M^+ \subseteq M^\bullet$.

Also note that, if M is finite, $core(M)$ is non-empty and connected wrt M, then $M^\bullet$ is a closed region. [$M^\bullet$ is finite since M is finite. $M^\bullet$ is closed and therefore complete. $core(M^\bullet) = core(M)$ is a non-empty maximal connected subset of $core(M)$. Suppose $p \in core(M^\bullet) \cap I(c)$. If $c = p$ then $c \in M^\bullet$ so assume $c \neq p$. We have $p \in core(M^\bullet)$, $c \in I(p)$ and $\dim(c) < \dim(p)$. Since $M^\bullet$ is closed, $c \in M^\bullet$. Therefore $M^\bullet$ is a component. Hence $M^\bullet$ is a closed component.]

If $M \subseteq S$, $\emptyset \neq core(M)$ is connected *wrt* M, then $M^\bullet$ is a closed component containing M. [Note that $M \subseteq M^\bullet$ and $core(M^\bullet) = core(M)$ is a non-empty maximal connected subset of $core(M)$. Since $M^\bullet$ is closed, it is also complete. Assume $p \in core(M^\bullet)$, $c \in M^\bullet$, and $c \in I(p)$. If $p = c$, then $c \in M^\bullet$, otherwise $\dim(c) < \dim(p)$. Since $p \in M^\bullet$ and $M^\bullet$ is closed, we have $c \in M^\bullet$. Thus $M^\bullet$ is a component. Therefore $M^\bullet$ is a closed component containing M.]

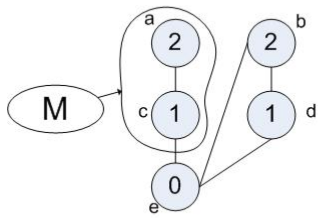

Fig. 6. A rooted, complete, finite $M \backslash \emptyset \neq core(M)$ which is connected wrt M, but $M^\bullet$ is not a rooted component of M.

Figure 6 shows that there exists a rooted, complete, finite, nonempty set $M \neq core(M)$ which is connected wrt M, but $M^\bullet$ is not a rooted component of M. For this, let G be defined by the diagram and let $M = \{a, c\}$. M is rooted, finite, and complete. Note $M^+ = M$ and $M^\bullet = \{a, c, e\}$. We have $e \in M^\bullet$ but $core(M^\bullet) = \{a\}$. Thus $M^\bullet$ is not rooted since $e \in M^\bullet$ and $core(M^\bullet) \cap I(e) = \emptyset$.

8 Open and Closed Regions; Complete Sets

If $M \subseteq S$ is finite then M is a closed region iff

(i) $\emptyset \neq core(M)$ is a non-empty, maximal connected subset of $core(M)$.
(ii) For all $c \in M$, $c' \in I(c)$ and $\dim(c') < \dim(c)$ it follows that $c' \in M$.

For showing this, assume at first that M is a closed region. Then Property (i) follows from the fact that M is a component. Property (ii) follows from the fact that M is closed. – Now assume that M satisfies Properties (i) and (ii). From Property (ii), M is closed and hence complete. - Assume p is a principal node of M and $c \in I(p)$. If $c = p$, then $c \in M$. So assume $s \neq p$. We have

$p \in M, c \in I(p)$, and $\dim(c) < \dim(p)$. Since M is closed, this implies $c \in M$. It follows that M is a closed region.

Corollary 29 *Let M be a finite subset of S where $G = [S, I, \dim]$ is an n-incidence pseudograph. Then*

- (i) *M is an open region of G iff*
  - (i.1) *$core(M)$ is non-empty and connected,*
  - (i.2) *$core(M) \cap I(c) \neq \emptyset \Rightarrow c \in M$, and*
  - (i.3) *if $\dim(c) < n$, then $c \in M \Leftrightarrow G(c) \subseteq M$.*
- (ii) *If G is monotonic then, M is an open region of G iff*
  - (ii.1) *$core(M)$ is non-empty and connected,*
  - (i.2) *$core(M) \cap I(c) \neq \emptyset \Rightarrow c \in M$, and*
  - (ii.3) *if $\dim(c) < n$, then $c \in M \Leftrightarrow core(S) \cap I(c) \subseteq M$.*

PROOF. Assume M is an open region of G. Properties (i.1) and (i.2) follow directly from the fact that M is a component. To prove Property (i.3), first assume $\dim(c) < n$ and $c \in M$ and $b \in G(c)$. Thus $b \in I(c)$ and $\dim(c) < \dim(b)$. Since M is open by Lemma 18, $b \in M$. Thus $G(c) \subseteq M$.

Next assume $\dim(c) < n$ and $G(c) \subseteq M$. There exists a $p \in core(S) \cap I(c)$. Thus $c \in I(p)$ and $\dim(c) < \dim(p)$ which implies $p \in G(c) \subseteq M$. Hence $p \in M$. Since M is closed this implies $c \in M$. Therefore Property (i.3) is satisfied by M.

Assume M is a subset satisfying Properties (i.1), (i.2), and (i.3). To show M is open assume $c \in M$, $c' \in I(c)$, and $\dim(c') > \dim(c)$. By Property (i.3), $c' \in M$. Thus, by Lemma 18, M is open.

To show that M is complete suppose $c \in M^+ \backslash M$. Thus, by Lemma 6, $G(c) \subseteq M_{i+1}$ where $i = \dim(c)$. It follows, by Property (i.3), that $c \in M$. This contradiction establishes $M^+ = M$ and hence M is complete. Thus we have that M is an open region. Therefore we have shown (i).

To prove (ii) assume G is monotonic.

Assume G is open. Then, by (i), G satisfies (ii.1), (ii.2), and (i.3). Suppose $dim(c) < n$. Assume $c \in M$ and $p \in core(S) \cap I(c)$. Thus $p \in G(c)$. By (i.3), $p \in M$. Thus $core(S) \cap I(c) \subseteq M$. Assume $core(S) \cap I(c) \subseteq M$ and let $b \in G(c)$. Thus $b \in I(c)$ and $dim(c) < dim(b)$. There exists $p \in core(S) \cap I(b)$. We have $c \in I(b), b \in I(p)$, and $dim(c) < dim(b) \leq dim(p$. Since G is monotonic, we have $c \in I(p)$. Hence $p \in core(S) \cap I(c)$ which, by assumptions, implies $p \in M$. Thus $p \in core(M) \cap I(b)$ and thus, by (i.2), $b \in M$. Hence $G(c) \subseteq M$ and thus, by (i.3), $c \in M$. Therefore M satisfies (ii.1), (ii.2), and (ii.3).

Assume G satisfies (ii.1), (ii.2), and (ii.3). Thus G satisfies (i.1) and (i.2). Suppose $dim(c) < n$. Assume $c \in M$ and $b \in G(c)$. There exists $p \in core(S) \cap I(b)$. We have $c \in I(b), b \in I(p)$, and $dim(c) < dim(b) \leq dim(p)$. Since G is monotonic $c \in I(p)$. Hence $p \in core(S) \cap I(c)$. In follows by (iii.3), $p \in M$. Since $p \in core(M) \cap I(b)$, we have by (ii.2), $b \in M$. Therefore $G(c) \subseteq M$.

Assume $G(c) \subseteq M$. Since $core(S) \cap I(c) \subseteq G(c)$, it follows from (ii.3), that $c \in M$. Therefore G satisfies (i.1), (i.2), and (i.3) and thus is open. $\square$

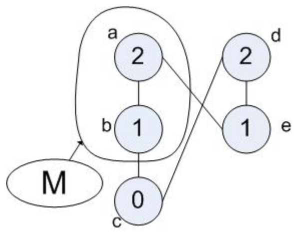

Fig. 7. A finite set M which is not a component.

Figure 7 shows that there exists an M which is finite, $core(M) \neq \emptyset$, connected, open, complete, and satisfies

$$\dim(c) < ind(G) \Rightarrow [c \in M \Leftrightarrow D(c) \subseteq M]$$

but is not a component as it fails to satisfy $core(M) \cap I(c) \neq \emptyset \Rightarrow c \in M$.

$c \in S$ is said to be *invalid wrt* M iff $c \notin M \wedge M \cap I(c) \neq \emptyset$. The following definition provides an alternative to the definition of a border as given above:

Definition 30 The set of all nodes invalid wrt M is called the *boundary of* M, denoted by $bd(M)$.

Theorem 31 (i) *If M is closed, then $bd(M) = \emptyset$.*
(ii) *If $bd(M) = \emptyset$, then M is complete.*

PROOF. Property (i): Assume M is closed and $c \in bd(M)$. Thus $c \notin M$ and there exists a $p \in core(M) \cap I(c)$. Hence $p \in I(c)$, $p \in M$, and $\dim(c) < \dim(p)$. Since M is closed this implies $c \in M$ but $c \notin M$. Therefore $bd(M) = \emptyset$.

Property (ii): Suppose M is not complete. Thus there exists a $c \in M^+ \backslash M$. Since there exists a principal node $p \in I(c)$, we have by Proposition 9, $p \in core(M) \cap I(c)$. Since $c \notin M$ this implies $c \in bd(M)$. But $bd(M) = \emptyset$. Therefore M is complete. $\square$

Obviously, this shows that any closed set is also complete. – The following are some technical specifications, needed in the following auxiliary considerations.

A node c is an *upward rooted point* of a set M iff $c \in M$ and there exists a descendence path $\{p_0, \ldots, p_k\}$ with

$$c = p_0 \wedge p_k \in core(M) \wedge \forall i \, (0 \leq i \leq k \Rightarrow p_i \in M)$$

The set of all upward rooted points of M is denoted by $URP(M)$.

A node c is a *downward exit point* of M iff $c \notin M$ and there exists a $c' \in M \cap I(c)$ with $\dim(c) < \dim(c')$. The set of all downward exit points of M is denoted by $DXP(M)$.

A node c is an *upward exit point* of M iff $c \notin M$ and there exists a $c' \in M \cap I(c)$ with $\dim(c) > \dim(c')$. The set of all upward exit points of M is denoted by $UXP(M)$.

(A1) M is closed iff $DXP(M) = \emptyset$. [Assume M is closed and suppose there exists a $c \in DXP(M)$. Thus $c \notin M$ and there exists a $c' \in M \cap I(c)$ such that $\dim(c) < \dim(c')$. Since M is closed this implies $c \in M$. This contradiction establishes $DXP(M) = \emptyset$.

Assume $DXP(M) = \emptyset$ and let $c \in M, c' \in I(c)$ such that $\dim(c') < \dim(c)$. Since $DXP(M) = \emptyset$, this implies $c' \in M$. Therefore M is closed.]

(A2) M is open iff $UXP(M) = \emptyset$. [Assume M is open and $c \in UXP(M)$. This implies $c \notin M$ and there exists a $c' \in M \cap I(c)$ such that $\dim(c') > \dim(c)$. But M is open which implies $c \in M$. Therefore $UXP(M) = \emptyset$.

Assume $UXP(M) = \emptyset$. Let $c \in M \wedge c' \in I(c)$ such that $\dim(c') > \dim(c)$. Since $UXP(M) = \emptyset$, this implies $c' \in M$. Therefore M is open.]

(A3) M is complete iff $DXP(M) \subseteq URP(\overline{M})$. [Assume M is complete and let $c \in DXP(M)$. Thus $c \notin M$ and there exists a vertex $b \in M \cap I(c)$ such that $\dim(c) < \dim(b)$. Thus $b \in G(c)$ and, since M is complete, $G(c) \neq \emptyset$, and $c \notin M = M^+$, we must have $G(c) \nsubseteq M$. Hence there exists a vertex $c' \in I(c) \cap \overline{M}$ such that $\dim(c') > \dim(c)$. Choose $p_0 = c$ and $p_1 = c'$.

Assume $p_0, \ldots, p_i$ have been chosen for some $i \geq 1$ satisfying for all j, $i \leq j \leq i$, $p_j \in I(p_{j-1}) \cap \overline{M}$ and $\dim(p_j) > \dim(p_{j-1})$. If $\dim(p_i) = \mathit{ind}(G)$, then we stop. Otherwise, since $p_i \in \overline{M}$ and M is complete we have $p_i \notin M^+$. Since $\dim(p_i) < \mathit{ind}(G)$, $G(p_i) \neq \emptyset$. Thus we must have $G(p_i) \nsubseteq M$. Hence there exists a $p_{i+1} \in G(p_i) \backslash M$. Then we have $p_{i+1} \in I(p_i) \cap \overline{M}$ and $\dim(p_{i+1}) > \dim(p_i)$. This process will eventually end. We choose k to be the final i and we end up with a descendence path $\{p_0, \ldots, p_k\}$ with $p_0 = c \wedge p_k \in \mathit{core}(\overline{M})$. Thus $c \in URP(\overline{M})$.

Assume $DXP(M) \subseteq URP(\overline{M})$. Let $n = \mathit{ind}(G)$ and for $0 \leq i \leq n$, consider the statement that $\mathbb{P}(i) \equiv \nexists c \in M^+ \backslash M$ with $\dim(c) = i$.

Suppose $c \in M^+ \backslash M \wedge \dim(c) = n$. Thus $c \in \mathit{core}(M^+) = \mathit{core}(M) \subseteq M$. Therefore $\mathbb{P}(n)$ is true.

Assume $\mathbb{P}(j)$ is true for all j such that $i \leq j \leq n$ for some i for $0 < i \leq n$ and suppose $c \in M^+ \backslash M$ such that $\dim(c) = i - 1$. Thus $\emptyset \neq G(c) \subseteq M_i^+$.

Thus there exists a c$'$ $\in G(c)$ such that $c' \in I(c)$, $\dim(c') > \dim(c) = i-1$, and $c' \in M_i^+ \subseteq M^+$. By assumption, since $i \leq \dim(c') \leq n$, $c' \notin M^+ \backslash M$. Since $c' \in M^+$, we must have $c' \in M$. Hence $c \in DXP(M) \subseteq URP(\overline{M})$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $c = p_0$, $\dim(p_k) = n$, and $p_j \notin M$ for all j satisfying $0 \leq j \leq k$. Since $\dim(c) < n$, $k > 0$. Thus $p_1 \in I(c)$, $\dim(p_1) > \dim(c)$, and $p_1 \notin M$. However, $p_1 \in G(c) \subseteq M^+$ which implies $p_1 \in M^+ \backslash M$ and $i \leq \dim(p_1) \leq n$. This contradiction establishes $\mathbb{P}(i-1)$. Therefore $M^+ \backslash M = \emptyset$ and therefore M is complete.]

(A4) M is open and complete iff $UXP(M) = \emptyset$ and $DXP(M) \subseteq URP(\overline{M})$

9 Partially Open Sets

For $i \in \mathbb{N}$ and $c \in S$, we define that

$$L_i(c) = \{c' \in I(c) : \dim(c') \leq i\}$$
$$L(c) = \{c' \in I(c) : \dim(c') < \dim(c)\}$$

From these definitions it follows that $I(c) = L(c) \cup cG(c)$, and, if $\dim(c) = i$, then $L(c) = L_{i-1}(c)$. [For the latter, note that $L(c) = \{c' \in I(c) : \dim(c' < \dim(c)\} = \{c' \in (c) : \dim(c' <= i-1\} = L_{i-1}(c)$.]

Let $M \subseteq S$ and $n = ind(G)$. For $0 \leq i \leq n$ we define M^- recursively by

$$M_0^- = M$$
$$M_{i+1}^- = M_i^- \cup \{c \in S : \dim(c) = i+1 \wedge \emptyset \neq L(c) \subseteq M_i^-\}$$

We define $M^- = \bigcup_{i=0}^n M_i^-$.

Definition 32 M is *partially open* iff $M = M^-$

If M is open, then M is partially open. [Let $n = ind(G)$ and suppose M is open. We claim $M_i^- \subseteq M$ for $0 \leq i \leq n$. – Since $M_0^- = M$, the claim is true for $i = 0$. Assume $M_i^- \subseteq M$ for some i, $0 \leq i < n$ and let $c \in M_{i+1}^-$. If $c \in M$ we are done so assume $c \notin M$. Thus we have $\dim(c) = i+1$ and $\emptyset \neq L(c) \subseteq M_i^-$. Thus there exists a $c' \in L(c)$ such that $c' \in M_i^-$, $c' \in I(c)$, and $\dim(c') < \dim(c)$. By assumption, $M_i^- \subseteq M$ and so $c' \in M$. Since M is open this implies $c \in M$. Therefore M is partially open.]

Lemma 33 *If* $n = ind(G)$ *and* $M \subseteq S$, *then*

(1) $M_i^- \subseteq M_j$, *for* $0 \leq i \leq j \leq n$
(2) $M_n^- = \bigcup_{i=0}^n M_i^-$
(3) if $0 < i \leq n$ *then* $c \in M_i^- \backslash M_{i-1}^- \Leftrightarrow \dim(c) = i \wedge \emptyset \neq L(c) \subseteq M_{i-1}^- \wedge c \notin M$
(4) if $\dim(c) = i \wedge c \in M^- \backslash M$ *then* $i > 0 \wedge c \in M_i^- \wedge \emptyset \neq L(c) \subseteq M_{i-1}^-$

PROOF. Properties (1), (2), and (3) follow immediately from the definitions. To prove Property (4) let $c \in M^- \backslash M$. Let k be the smallest natural number such that $c \in M_k^-$. Since $c \notin M = M_0^-$, $k > 0$ and $c \in M_k^- \backslash M_{k-1}^-$. Thus $i = k > 0$ and $\emptyset \neq L(c) \subseteq M_{i-1}^-$. □

Theorem 34 *For $M \subseteq S$, M^- is the smallest subset of S satisfying:*

(1) $M \subseteq M^-$
(2) if $\emptyset \neq L(c) \subseteq M^-$ then $c \in M^-$.

PROOF. Property (1) follows from $M = M_0 \subseteq M^-$. To prove Property (2), assume $\emptyset \neq L(c) \subseteq M^-$. If $c \in M$, then $c \in M^-$ so assume $c \notin M$. Thus, by Lemma 33, $\emptyset \neq L(c) \subseteq M_{i-1}^-$ and $c \in M_i^- \backslash M_{i-1}^-$. Hence $c \in M^-$. Therefore M satisfies Properties (1) and (2).

Suppose A satisfies Properties (1) and (2). Let $c \in M^-$. If $c \in M$, then $c \in A$ since A satisfies Property (1). Assume $c \notin M$. Thus, by Lemma 33, $c \in M_i^- \backslash M_{i-1}^-$ and $\emptyset \neq L(c) \subseteq M_{i-1}^-$ where $i = \dim(c)$. Hence $c \in M^- \backslash M$. We claim that this is sufficient to show $c \in A$.

Let $\mathbb{P}(i)$ be the statement "If $c \in M_i^- \backslash M \wedge \dim(c) = i$, then $c \in A$". Since $M_0^- = M$, $\mathbb{P}(0)$ is true (vacuously.)

Let $n = ind(G)$ and assume $\mathbb{P}(j)$ is true for all $0 \leq j \leq i$ for some $i, 0 \leq i < n$. Let $c \in M_{i+1}^- \backslash M$ such that $\dim(c) = i + 1$. It follows from Lemma 33 that $\emptyset \neq L(c) \subseteq M_i^-$. Let $c' \in L(c)$ and $k = \dim(c')$. Thus $c' \in M_i^-$ and $k \leq \dim(c) - 1 = i$. If $c' \in M$ then $c' \in A$ so assume $c' \notin M$. It follows that $c' \in M_k^- \backslash M$. Since $0 \leq k \leq i$, and the assumption $\mathbb{P}(k)$ is true, we have that $c' \in A$. Thus $\emptyset \neq L(c) \subseteq A$. Since A satisfies Property (2), $c \in A$. Hence $\mathbb{P}(i)$ is true for all $0 \leq i \leq n$. Therefore $M^- \subseteq A$. □

For $M \subseteq S$, and $i \in \mathbb{N}$ we define

$$\mathbb{O}(M, i) = \{c \in S : \exists \text{ descendence path } \{p_0, \dots, p_i\} \text{ with } c = p_0 \wedge p_0 \in M\}$$
$$\mathbb{O}(M) = \bigcup_{i=0}^{\infty} \mathbb{O}(M, i)$$

Lemma 35 *If A is open and $M \subseteq A$, then*

(i) $\mathbb{O}(M, i) \subseteq A$*, for all* $i \in \mathbb{N}$
(ii) $\mathbb{O}(M) \subseteq A$

PROOF. Property (ii) follows from Property (i) since $\mathbb{O}(M) = \bigcup_{i=0}^{\infty} \mathbb{O}(M, i)$. Let $n = ind(G)$. Note that $\mathbb{O}(M, i) = \emptyset$, $\forall i > n$. We will prove Property (i) by induction. Since $\mathbb{O}(M, 0) = M$ and $M \subset A$, we have $\mathbb{O}(M, 0) \subseteq A$.

Assume $\mathbb{O}(M, i) \subseteq A$ and let $c \in \mathbb{O}(M, i + 1)$ for some i such that $0 \leq i < n$. By definition there exists a descendence path $\{p_0, \dots, p_{i+1}\}$ with $c = p_{i+1}$ and $p_0 \in M$. Note $\{p_0, \dots, p_i\}$ is a descendence path with $p_0 \in M$ and thus $p_i \in \mathbb{O}(M, i)$. By assumption this implies $p_i \in A$. We have $p_i \in A$, $p_i \in I(c)$, and $\dim(p_i) < \dim(c)$. Since A is open this implies $c \in A$. □

Corollary 36 *If A is open, $M \subseteq A$, and if $\{p_0, \dots, p_k\}$ is a descendence path with $p_0 \in M$, then $p_k \in A$*

Theorem 37 *If M is finite, then $\mathbb{O}(M)$ is finite*

PROOF. Assume M is finite. Thus $\mathbb{O}(M, 0) = M$ is finite. Let $n = ind(G)$ and assume $\mathbb{O}(M, i)$ is finite for some i such that $0 \leq i < n$. We show that

$$\mathbb{O}(M, i+1) = \{c \in S : \exists c' \in \mathbb{O}(M, i) \text{ with } c' \in I(c) \wedge \dim(c') < \dim(c)\}$$

Let $A = \{c \in S : \exists c' \in \mathbb{O}(M, i) \text{ with } c' \in I(c) \wedge \dim(c') < \dim(c)\}$ and let $c \in \mathbb{O}(M, i+1)$. By definition there exists a descendence path $\{p_0, \dots, p_{i+1}\}$ such that $p_0 \in M$ and $c = p_{i+1}$. Let $c' = p_i$. Note $\{p_0, \dots, p_i\}$ is a descendence path with $p_0 \in M \wedge c' = p_i$. Thus $c' \in \mathbb{O}(M, i)$, $c' \in I(c)$, and $\dim(c') < \dim(c)$. Thus $c \in A$. Therefore $\mathbb{O}(M) \subseteq A$.

Now let $c \in A$. Thus there exists a $c' \in \mathbb{O}(M, i)$ with $c' \in I(c)$ and $\dim(c') < \dim(c)$. Let $\{p_0, \dots, p_i\}$ be a descendence path with $p_0 \in M \wedge c' = p_i$. Let $s_{i+1} = c$ and let $s_j = p_j$ for $0 \leq j \leq i$. Note that $\{s_0, \dots, s_{i+1}\}$ is a descendence path with $s_0 \in M$ and $c = s_{i+1}$. Hence $c \in \mathbb{O}(M, i+1)$ and therefore $\mathbb{O}(M, i+1) = A$.

Note if $c \in \mathbb{O}(M, i+1)$, then there exists a $c' \in \mathbb{O}(M, i)$ such that $c \in I(c')$. Thus $\mathbb{O}(M, i+1) \subseteq \bigcup\{I(c') : c' \in \mathbb{O}(M, i)\}$, which is a finite union of finite sets. $\square$

It follows that M^- is finite if M is finite. [Note that $M^- \subseteq \mathbb{O}(M)$.]

If G is monotonic, then $\mathbb{O}(M) = M \cup \mathbb{O}(M, 1)$. [Assume G is monotonic and let $A = M \cup \mathbb{O}(M, 1)$ and $n = ind(G)$. We claim $\mathbb{O}(M, i) \subseteq \mathbb{O}(M, 1)$ for $1 \leq i \leq n$. Clearly it is true for $i = 1$. Assume $\mathbb{O}(M, i) \subseteq \mathbb{O}(M, 1)$ for some i such that $1 \leq i < n$ and let $c \in \mathbb{O}(M, i+1)$. Thus there exists a descendence path $\{p_0, \dots, p_{i+1}\}$ such that $p_0 \in M$ and $c = p_{i+1}$. Since $\{p_0, \dots, p_i\}$ is a descendence path with $p_0 \in M$, we have $p_i \in \mathbb{O}(M, i)$ which, by assumption implies $p_i \in \mathbb{O}(M, 1)$. Thus there exists a descendence path $\{a, p_i\}$ with $a \in M$. We have $a \in I(p_i)$, $\dim(a) < \dim(p_i)$, $p_i \in I(p_{i+1})$, and $\dim(p_i) < \dim(p_{i+1})$. Since G is monotonic, $a \in I(p_{i+1})$ and thus $\{a, p_{i+1}\}$ is a descendence path with $a \in M$ which implies $c = p_{i+1} \in \mathbb{O}(M, 1)$. Therefore $M \cup \mathbb{O}(M, 1) = \bigcup_{i=0}^{n} \mathbb{O}(M, i) = \mathbb{O}(M)$.]

Theorem 38 $\mathbb{O}(M)$ *is the smallest open set containing M.*

PROOF. $M = \mathbb{O}(M, 0) \subseteq \mathbb{O}(M)$. To show $\mathbb{O}(M)$ is open, let $c \in \mathbb{O}(M)$ and $c' \in I(c)$ such that $\dim(c') > \dim(c)$. Thus there exists a descendence path $\{p_0, \dots, p_k\}$ such that $p_0 \in M$ and $c = p_k$. Define $p_{k+1} = c'$ then $\{p_0, \dots, p_{k+1}\}$ is a descendence path with $p_0 \in M$ and $c' = p_{k+1}$. Thus $c' \in \mathbb{O}(M)$. Therefore $\mathbb{O}(M)$ is open.

Suppose $M \subseteq A$ for some open set A. By Lemma 35, $\mathbb{O}(M) \subseteq A$. $\square$

The last theorem defines a dual to the topological closure of a set M, which is normally not available in other topologies.

10 0-Rooted Sets

Finally we reverse the roles of principal and 0-dimensional nodes. For example, in picture analysis it might be of interest to focus on grid vertices (corners of grid cubes, end points of grid edges, and so forth) rather than on pixel or voxel, identified by principal nodes. This will also support studies of "partially open" sets (i.e., sets which are typically not studied in topological papers related to binary picture processing).

For $M \subseteq S$ we define $leaves(M) = \{c \in M : \dim(c) = 0\}$. A node $c \in M$ is said to be *0-rooted* in M iff $I(c) \cap leaves(M) \neq \emptyset$. The set of all 0-rooted nodes in M is denoted by *0-Rooted(M)*.

Let *0-Unrooted(M)* $= M \setminus$ 0-Rooted(M). If $M =$ 0-Rooted(M), then we say M is *0-rooted*.

Definition 39 If S is 0-rooted, then we also say that G is *0-rooted*.

Corollary 40 $leaves(M^-) = leaves(M)$

PROOF. Let $c \in leaves(M^-)$. Thus $\dim(c) = 0 \wedge c \in M^-$. Suppose $c \notin M$. Then, by Lemma 33, $\dim(c) > 0$. Thus $c \in M$ which implies $c \in leaves(M)$. Therefore $leaves(M^-) \subseteq leaves(M)$.

Let $c \in leaves(M)$. Thus $c \in M \wedge \dim(c) = 0$ which implies $c \in M^- \wedge \dim(c) = 0$. Thus $c \in leaves(M^-)$. Therefore $leaves(M^-) = leaves(M)$. $\square$

$leaves(\mathbb{O}(M)) = leaves(M)$: Since $m \subseteq \mathbb{O}(M)$ it follows that $leaves(M) \subseteq leaves(\mathbb{O}(M))$. Let $c \in leaves(\mathbb{O}(M))$. Thus $c \in \mathbb{O}(M)$ and $\dim(c) = 0$. This implies there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $p_0 \in M$ and $c = p_k$. Note that if $k > 0$, $p_{k-1} \in I(c)$ and $\dim(p_{k-1}) < \dim(c)$. But $\dim(c) = 0$ and thus $k = 0$ and $c = p_0 \in M$. Hence $c \in leaves(M)$. Therefore $leaves(\mathbb{O}(M)) = leaves(M)$.

$M^- \subseteq \mathbb{O}(M)$: $\mathbb{O}(M)$ contains M and is open and therefore partially open. Since M^- is the smallest partially open set containing M, $M^- \subseteq \mathbb{O}(M)$.

Lemma 41 *If* $c \in M^- \backslash M$ *and* $b \in leaves(S) \cap I(c)$*, then* $b \in M$*.*

PROOF. Let $c \in M^- \backslash M$ and $b \in leaves(S) \cap I(c)$. Let $i = \dim(c)$. Thus, by Lemma 33, $i > 0 \wedge c \in M_i^- \backslash M$ and $\emptyset \neq L(c) \subseteq M_{i-1}^-$. We have $b \in I(c)$ and $\dim(b) < \dim(c)$. Thus $b \in L(c)$ and hence $b \in M_{i-1}^-$. This implies $b \in M^-$. Since $\dim(b) = 0$, $b \in leaves(M^-) = leaves(M)$. Therefore $b \in M$. $\square$

If G is 0-rooted and $c \in M^- \backslash M$, then $leaves(M) \cap I(c) \neq \emptyset$. [Let $c \in M^- \backslash M$. Since G is 0-rooted, there exists a $b \in leaves(S) \cap I(c)$. By theorem 41, this implies $b \in M$ and therefore $leaves(M) \cap I(c) \neq \emptyset$.]

If M is 0-rooted, then M^- is 0-rooted. [Assume M is 0-rooted and let $c \in M^-$. If $c \in M$, then $leaves(M) \cap I(c) \neq \emptyset$. Since $leaves(M) = leaves(M^-)$, this implies $leaves(M^-) \cap I(c) \neq \emptyset$.]

If $M \neq \emptyset$ and M is 0-rooted, then $leaves(M) \neq \emptyset$. [Since $M \neq \emptyset$ there exists a $c \in M$. Since M is 0-rooted, $leaves(M) \cap I(c) \neq \emptyset$. Therefore $leaves(M) \neq \emptyset$.]

Theorem 42 *If G is 0-rooted and M is partially open, then 0-Rooted(M) is partially open and 0-rooted.*

PROOF. Let $A = $ 0-Rooted(M). Note that $leaves(A) = leaves(M)$. Suppose $c \in A$. Thus $c \in M$ and $leaves(M) \cap I(c) \neq \emptyset$ and so $leaves(A) \cap I(c) \neq \emptyset$. Therefore A is 0-rooted.

To show that A is partially open, suppose there exists a $c \in A^- \backslash A$. Since $A \subseteq M$, we have $A^- \subseteq M^-$. Since M is partially open this implies $A^- \subseteq M$ and hence $c \in M \backslash A$ which implies $leaves(M) \cap I(c) = \emptyset$. Let $i = \dim(c)$. Since G is 0-rooted, there exists a $b \in leaves(S) \cap I(c)$ and since $c \in A^- \backslash A$, we have, by Proposition 41, $b \in A$ and hence $leaves(A) \cap I(c) \neq \emptyset$. But, since $leaves(A) = leaves(B)$, this implies $leaves(M) \cap I(c) \neq \emptyset$. This contradiction established that A is partially open. □

11 0-Components and 0-Regions

$C \subseteq M$ is a *0-component of M* iff

(1) $leaves(A)$ form a non-empty maximal connected (wrt M) subset of $leaves(M)$,
(2) if $b \in leaves(C) \wedge c \in M \wedge c \in I(b)$, then $c \in C$, and
(3) C is partially open.

A finite 0-component of M is called a *0-region of M*.

Definition 43 If M is a 0-component of M, then we call M a *0-region.*

Let M be partially open, 0-rooted, and $leaves(M)$ is connected. If C is a 0-component of M, then $C = M$. [Since C is a 0-component of M, $C \subseteq M$. Since $leaves(M)$ is connected and $leaves(C)$ is a maximal connected subset of $leaves(M)$, we must have $leaves(C) = leaves(M)$. Let $c \in M$. Since M is 0-rooted, $I(c) \cap leaves(M) \neq \emptyset$ and thus $I(c) \cap leaves(C) \neq \emptyset$. Since C is a 0-component of M this implies $c \in C$. Therefore $M = C$.]

Lemma 44 *If G is 0-rooted and C is a 0-component of M, then 0-Rooted(C) is a 0-rooted component of M.*

PROOF. Let R =0-Rooted(C). Note that $leaves(R) = leaves(C)$ is a non-empty maximal connected subset of $leaves(M)$ since C is a 0-component of M. Clearly R is 0-rooted.

Let $c \in M$ such that $leaves(R) \cap I(c) \neq \emptyset$. Thus $c \in M$ and $leaves(C) \cap I(c) \neq \emptyset$. Since C is a 0-component this implies $c \in C$. Since $leaves(C) \cap I(c) \neq \emptyset$ we have $c \in R$.

To show R is 0-complete assume there exists a $c \in R^{-} \backslash R$. Since G is 0-rooted there exists a $b \in leaves(S) \cap I(c)$. Thus, by Proposition 41, $b \in R$ and thus $leaves(R) \cap I(c) \neq \emptyset$. Since $leaves(R) = leaves(C)$, $leaves(C) \cap I(c) \neq \emptyset$. Since $R \subseteq C, R^{-} \subseteq C^{-} = C, c \in C$, and $c \notin R$, we have $leaves(C) \cap I(c) \neq \emptyset$. This contradiction establishes that R is 0-complete. Therefore R is a 0-rooted 0-component of M. $\square$

Lemma 45 *If G is 0-rooted, M is partially open, and $b \in leaves(M)$, then M has a unique 0-rooted 0-component C containing b. Furthermore $C = leaves(C) \cup \{c \in M : leaves(C) \cap I(c) \neq \emptyset\}$*

PROOF. Let $b \in leaves(M)$ for M partially open. Let $A = \{c \in leaves(M) : c \text{ and } b \text{ are connected } wrtM\}$. Note A is a non-empty maximal connected subset of $leaves(M)$. Let $C = A \cup \{c \in M : A \cap I(c) \neq \emptyset\}$. Note $C \subseteq M$ and $leaves(C) = A$. Thus $leaves(C)$ is a non-empty maximal connected subset of $leaves(M)$.

To show C is 0-complete suppose there exists a $c \in C^{-} \backslash C$ which implies $A \cap I(c) = \emptyset$. Since G is 0-rooted there exists a $p \in leaves(S) \cap I(c)$ which, by theorem 41,implies $p \in leaves(C) \cap I(c)$. This contradiction establishes that C is 0-complete.

Let $p \in leaves(C)$ and $c \in M$ such that and $c \in I(p)$. Thus $c \in M$ and $A \cap I(c) \neq \emptyset$ which implies $c \in C$. Therefore C is a 0-component of M. Clearly C is 0-rooted. Therefore C is a 0-rooted 0-component of M containing b.

To show that C is unique, assume R is a 0-rooted 0-component of M containing B. Since both C and R are 0-rooted and contain b, there exists a $c \in leaves(C)$ and an $r \in leaves(R)$ such that $c \in I(b)$ and $r \in I(c)$. Thus $leaves(R)$ and $leaves(C)$ are connected in M by b and since $leaves(R)$ and $leaves(C)$ are both maximal connected subsets of $leaves(M)$, we must have $leaves(R) = leaves(C) = A$.

Let $c \in C$ and hence $c \in M$. If $c \in A$, then $c \in R$ so assume $c \notin A$ which implies $A \cap I(c) \neq \emptyset$ and hence $leaves(R) \cap I(c) \neq \emptyset$. Since R is a 0-component of M this implies $c \in R$. Thus $C \subseteq R$.

Let $c \in R$ which implies $c \in M$. Since R is 0-rooted there exists a $p \in leaves(R) \cap I(c)$. Thus $c \in M$ and $A \cap I(c) \neq \emptyset$ which implies $c \in C$. Therefore $C = R$. $\square$

Lemma 46 *If G is 0-rooted and M is partially open and 0-rooted, then the 0-rooted 0-components of M partition M*

PROOF. For each $b \in leaves(M)$, let C_b be the unique 0-rooted 0-component of M containing b. Recall $C_b = leaves(C_b) \cup \{c \in M : leaves(C_p) \cap I(c) \neq \emptyset\}$. Let $\mathbb{P} = \{C_b : b \in leaves(M)\}$. Suppose $a, b \in leaves(M)$ such that $C_a \cap C_b \neq \emptyset$. Let $c \in C_a \cap C_b$. Since C_a and C_b are 0-rooted there exists an $a' \in leaves(C_a) \cap I(c)$ and there exists a vertex $b' \in leaves(C_b) \cap I(c)$. We have that $[a, a', c, b', b]$ is a sequence of nodes in M each connected to the next and thus a and b are connected *wrt* M and hence $a \in C_b$ which implies $C_a = C_b$.

Let $c \in M$. Since M is 0-rooted, there exists $b \in leaves(M) \cap I(c)$. Thus $c \in C_b$ which implies $c \in \bigcup \mathbb{P}$. Since $\bigcup \mathbb{P} \subseteq M$ we have $M = \bigcup \mathbb{P}$. Therefore $\mathbb{P}$ partitions M. $\square$

Lemma 47 *If C is a 0-component of 0-Rooted(M), then C is a 0-rooted 0-component of M.*

PROOF. Let $K =$ 0-Rooted(M) and let C be a 0-component of K. Note that $leaves(K) = leaves(M)$ and thus $leaves(C)$ is a non-empty maximal connected subset of $leaves(M)$.

Assume $p \in leaves(K)$, $c \in K$, and $c \in I(p)$. Thus $leaves(M) \cap I(c) \neq \emptyset$ and hence $c \in K$. Since C is a 0-component of K we have $c \in C$. Since C is partially open, C is a 0-component of M. Since $C \subseteq$ 0-Rooted(M), we have $leaves(M) \cap I(c) \neq \emptyset$, for all $c \in C$. Therefore C is a 0-rooted 0-component of M. $\square$

Corollary 48 *If G is 0-rooted and M is partially open and not 0-rooted, then the set consisting of $0\text{–}Unrooted(M)$ along with the 0-rooted 0-components of M forms a partition of M.*

PROOF. Let $K = M \setminus$ 0-Unrooted$(M) =$ Rooted(M). By Lemma 44, K is 0-complete and 0-rooted. Let $\mathbb{P}$ be the collection of the 0-rooted 0-components of K along with $Unrooted(M)$. By Lemma 47, the 0-rooted 0-components of K are 0-rooted 0-components of M. By Lemma 46, K is the union of the 0-rooted 0-components of K (and hence of M.) Since $M =$ 0-Unrooted$(M) \cup K$, we have $M = \bigcup \mathbb{P}$. Since the 0-rooted 0-components of M are disjoint and distinct from each other and from 0-Unrooted(M), $\mathbb{P}$ partitions M. $\square$

We demonstrate the existence of some particular kinds of set by means of examples. There exists a finite M which is complete with $\overline{M}$ not partially open. For this, see $M = \{1, c\} = M^+$ and $\overline{M} \neq \overline{M}^- = \{a, b, c, d\}$ on the left in Figure 8.

There exists a finite M which is partially open with $\overline{M}$ not complete; see right of Figure 8: $M = \{b, c\} = M^-$, and $\overline{M} = \{a, d, e\} \neq \overline{M}^- = \{a, b, c, d, e\}$.

Fig. 8. Left: a finite set M which is complete, and $\overline{M}$ is not partially open. Right: a finite M which is partially open with $\overline{M}$ not complete.

There exists a finite M which is open (and hence partially open) with $M \neq \mathit{leaves}(M)^-$. See $M = \{a, b, d\} = M^-$ and $\mathit{leaves}(M)^- = \{d\} \neq M$ in Figure 9, left.

Fig. 9. Left: a finite set M which is open, with $M \neq \mathit{leaves}(M)^-$. Right: a finite set M which is closed, partially open, and for which M^∇ is not partially open.

If M is open, then M^∇ is partially open. [Assume M is open and $\emptyset \neq L(c) \subseteq M^\nabla$. To show $c \in M^\nabla$, let $b \in I(c)$. Note that there exists a $c' \in L(c)$ with $c' \in I(c)$ and $\dim(c')$. By assumption $c' \in M^\nabla$. Thus $I(c') \subseteq M$. Since $c' \in I(c)$ implies $c \in I(c')$, we have $c \in M$. – If $\dim(b) > \dim(c)$, then $b \in M$ since M is open. If $\dim(b) = \dim(c)$, then $b = c \in M$. If $\dim(b) < \dim(c)$, then $b \in L(c) \subseteq M^\nabla \subseteq M$. In all cases we have that $b \in M$. Hence $c \in M^\nabla$ and therefore M^∇ is partially open.]

There exists a closed, partially open M for which M^∇ is not partially open. Figure 9 shows such a set on the right, with $M = \{a, b, c\}$, $M^\nabla = \{c\}$ and $L(a) = M^\nabla$ but $a \notin M^\nabla$. Therefore M^∇ is not partially open.

If $M \neq \emptyset$ and $M^\nabla = \emptyset$ (i.e., $\delta M = M$), then M is not open or M is not closed. [Since $M \neq \emptyset$, there exists a $c \in M$. Since $c \notin M^\nabla$, there exists a $c' \in I(c) \backslash M$. If $\dim(c') > \dim(c)$, then M is not open. We cannot have $\dim(c') = \dim(c)$ which would imply $c' = c$ since $c \in M$ and $c' \notin M$. If $\dim(c') < \dim(c)$, then M is not closed.]

There exists an M which is partially open and closed (and therefore complete) with $M^\nabla = \emptyset$. See Figure 10, left.

There exists an M which is complete and open (and therefore partially open) with $M^\nabla = \emptyset$. See Figure 10, middle.

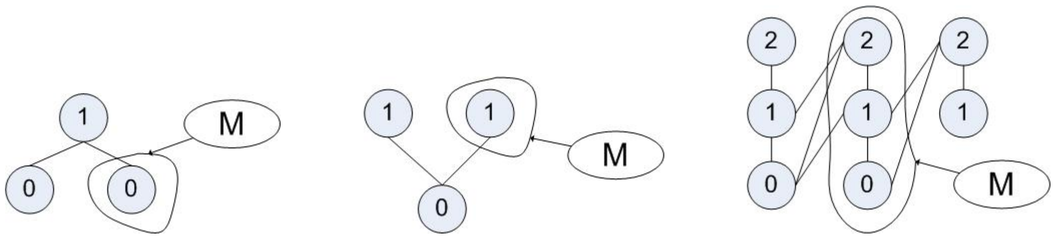

Fig. 10. Left: set M which is partially open and closed with $M^\nabla = \emptyset$. Middle: set M which is complete and open with $M^\nabla = \emptyset$. Right: set M which is partially open and complete, $core(M) \neq \emptyset$, $leaves(M) \neq \emptyset$, rooted and 0-rooted, with $M^\nabla = \emptyset$.

There exists an M which is partially open and complete, $core(M) \neq \emptyset$, $leaves(M) \neq \emptyset$, rooted and 0-rooted, with $M^\nabla = \emptyset$. See Figure 10, right.

A node c is a *downward 0-rooted point* of M iff $c \in M$ and there exists a descendence path $\{p_0, \ldots, p_k\}$ such that

$$p_0 \in leaves(M) \wedge c = p_k \wedge \forall i \,(\, 0 \leq i \leq k \;\rightarrow\; p_i \in M\,)$$

The set of all downward 0-rooted points of M is denoted by $DRP(M)$.

Theorem 49 *If G is 0-rooted, then M is partially open iff $UXP(M) \subseteq DRP(\overline{M})$.*

PROOF. Assume G is 0-rooted.

Assume M is partially open and let $c \in UXP(M)$. Thus $c \notin M$ and there exists a $b \in M$ such that $\dim(b) < \dim(c)$. Hence $L(c) \neq \emptyset$ and, since M is partially open, $L(c) \neq \emptyset$, and $c \notin M = M^-$, we must have $L(c) \neq M$. Thus there exists $c' \in I(c) \cap \overline{M}$ such that $\dim(c') < \dim(c)$. Chose $p_0 = c$ and $p_1 = c'$.

Assume $p_0, \ldots, p_i$ have been chosen for some $i \geq 1$, such that for all j such that $1 \leq j \leq i, p_j \in I(p_{j-1})$ and $\dim(p_j) < \dim(p_{j-1})$. If $\dim(p_i) = 0$ we set $k = i$ and stop. Otherwise, since G is 0-rooted and $\dim(p_i) > 0$, we have $L(p_i) \neq \emptyset$. Since $p_i \in \overline{M}$, $L(p_i) \neq \emptyset$,and M is partially open, we have $p_i \notin M^-$. Hence there exists a $p_{i+1} \in L(p_i) \backslash M$. Thus we have $p_{i+1} \in I(p_i) \cap \overline{M}$ and $\dim(p_{i+1}) < \dim(p_i)$. This process will eventually end.

Define $s_j = p_{k-j}$, for each j such that $0 \leq j \leq k$. Then $\{s_0, \ldots, p_k\}$ is a descendence path with $s_0 \in leaves(\overline{M})$ and $s_k = c$. Thus $c \in DRP(\overline{M})$ and therefore $UXP(M) \subseteq DRP(\overline{M})$.

Assume $UXP(M) \subseteq DRP(\overline{M})$. Let $n = ind(G)$ and for $0 \leq i \leq n$ consider the statement

$$\mathbb{P}(i) \quad \equiv \quad \nexists c \in \overline{M} \backslash M \text{ with } \dim(c) = i$$

Suppose $c \in M^{-}\backslash M$ and $\dim(c) = 0$. Thus $c \in leaves(M^{-}) = leaves(M) \subseteq M$. Therefore $\mathbb{P}(0)$ is true.

Assume $\mathbb{P}(j)$ is true for all j such that $0 \leq j \leq i$ for some i with $0 \leq i < n$ and suppose $c \in M^{-}\backslash M$ with $\dim(c) = i+1$. Thus $\emptyset \neq L(c) \subseteq M^{-}$. Hence there exists a $c' \in L(c)$ such that $c' \in I(c)$, $\dim(c') < \dim(c) = i+1$, and $c' \in M_i^{-} \subseteq M^{-}$. By assumption, since $\dim(c') \leq i$, $c' \notin M^{-}\backslash M$. Since $c') \in M^{-}$, we must have $c' \in M$. Hence $c \in DXP(M) \subseteq URP(\overline{M})$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $p_0 \in leaves(\overline{M})$, $p_k = c$, and $p_j \in \overline{M}$, for all j satisfying $0 \leq j \leq k$. Since $\dim(c') < \dim(c)$, $\dim(c) > 0$ and so $k > 0$. Thus $p_{k-1} \in I(c)$, $\dim(p_{k-1}) < \dim(c)$, and $p_{k-1} \notin M$. However, $p_{k-1} \in L(c) \subseteq M^{-}$, which implies $p_{k-1} \in M^{-}\backslash M$ and $\dim(p_{k-1}) = i$. Thus, by assumption $\mathbb{P}(\dim(p_{k-1}))$ is true which implies $p_{k-1} \notin M^{-}\backslash M$. This contradiction establishes that $\mathbb{P}(i+1)$ is true and therefore $M^{-}\backslash M = \emptyset$. Therefore M is partially open. $\square$

Corollary 50 *A set M is closed and partially open iff $DXP(M) = \emptyset$ and $UXP(M) \subseteq DRP(\overline{M})$.*

12 Concluding Remarks

This paper provided a comprehensive discussion of a topology on incidence pseudographs, as introduced by Klaus Voss in 1993, and further discussed by others in more recent years. (The references below only give a very limited account of such work; for an extensive bibliography see, for example, (7).) The paper also discussed (for the first time) especially partially open sets, as occurring in common (non-binary) digital picture analysis.

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